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Poisson structures on double Lie groups

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Abstract

Lie bialgebra structures are reviewed and investigated in terms of the double Lie algebra, of Manin- and Gauss-decompositions. The standard R-matrix in a Manin decomposition then gives rise to several Poisson structures on the correponding double group, which is investigated in great detail.

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1. Introduction

In [2] we described a wide class of symplectic structures on the cotangent bundle T^*G of a Lie group G by replacing the canonical momenta of actions of G on T^*G by arbitrary ones. This method also worked for principal bundles and allowed us to describe the notion of a Yang–Mills particle which carries a 'charge' given by spin-like variables, by means of Poisson reduction.

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In the latter half of this paper we consider 'deformations' of T^*G in the form of so called double Lie groups equipped with the analogs of the symplectic structure on T^*G, closely related to Poisson–Lie groups. Parts of the results may be found spread over different places, mainly in the unfortunately unpublished thesis of Lu [20], but also to some extend in [1,33], and others. Our presentation makes the double group the main object rather than Poisson–Lie groups, which makes the roles of G and G^* manifestly symmetric and contains all the information about G and G^* and all relations between them. All these are also associated to the theory of symplectic groupoids as 'deformed cotangent bundles' in general, and with mechanical systems based on Poisson symmetries as studied for instance in [23,36]. The explicit formulae from the second part have already found applications in [3].

The first half of this paper is devoted to the general setup: Recall that a Poisson–Lie group is a Lie group G with a Poisson structure $\Lambda \in \Gamma(\wedge^2 TG)$ such that the multiplication map $G \times G \to G$ is a morphism of the Poisson manifolds. The corresponding infinitesimal object, which determines a Poisson–Lie group up to a covering, is that of a Lie bialgebra, defined by V.G. Drinfeld. It is defined as a Lie algebra $(\mathfrak{g}, b = [\ ,\ ])$ together with the structure of a Lie algebra $(\mathfrak{g}^*, b' = [\ ,\ ])$ on the dual space $\mathfrak{g}^*$ such that the bracket b' defines a cocycle $b' : \mathfrak{g} \to \wedge^2\mathfrak{g}$ on $\mathfrak{g}$ with values in the $\mathfrak{g}$-module $\wedge^2\mathfrak{g}$. The brackets b, b' define the structure of a metrical Lie algebra on $\mathfrak{l} = \mathfrak{g} \oplus \mathfrak{g}^*$ with Manin decomposition. Recall that a metrical Lie algebra is a Lie algebra together with a non-degenerate ad-invariant bilinear symmetric form g (the metric), and that a Manin decomposition is a decomposition of a metrical Lie algebra into direct sum of two isotropic subalgebras. The metric g on $\mathfrak{l}$ is defined by the conditions that the subspaces $\mathfrak{g}$, $\mathfrak{g}^*$ are isotropic and the restriction of g on $\mathfrak{g} \times \mathfrak{g}^*$ is the natural pairing. Hence, there is a natural bijection between Poisson–Lie groups (up to a covering), bialgebras, and metrical Lie algebras with Manin decompositions. Remark that not every metrical Lie algebra admits a Manin decomposition [8]. We recall some basic constructions and facts on metrical Lie algebras in 2.4–2.7. A bivector $C \in \wedge^2\mathfrak{g}$ on a Lie algebra $\mathfrak{g}$ defines a cocycle

$$\partial C : \mathfrak{g} \to \wedge^2\mathfrak{g}, \quad X \mapsto \mathrm{ad}_X C.$$

Moreover, C defines a structure of a Lie algebra on $\mathfrak{g}^*$ if and only if the Schouten bracket $[C, C]$ is $\mathrm{ad}_{\mathfrak{g}}$ invariant. This condition is called the modified Yang–Baxter equation.

For a metrical Lie algebra $(\mathfrak{g}, g)$ a bivector C can be identified with an endomorphism $R = C \circ g$ (the 'R-matrix'). In terms of this endomorphism the modified Yang–Baxter equation (and other equations implying this) reduces to the generalized R-matrix equation (and some modifications of it), see (2.9). A Manin decomposition $\mathfrak{g} = \mathfrak{g}_+ \oplus \mathfrak{g}_-$ of a metrical Lie algebra $\mathfrak{g}$ provides a solution $R = \mathrm{pr}_+ - \mathrm{pr}_-$ of the R-matrix equation. More generally, we define a Gauss decomposition of a metrical Lie algebra $\mathfrak{g}$ as a decomposition $\mathfrak{g} = \mathfrak{g}_+ \oplus \mathfrak{g}^0 \oplus \mathfrak{g}_-$ of $\mathfrak{g}$ into a sum of subalgebras such that $\mathfrak{g}_+$, $\mathfrak{g}_-$ are isotropic and orthogonal to $\mathfrak{g}^0$. Any solution R^0 of the R-matrix equation (1-mYBE) on $\mathfrak{g}^0$, see 2.9, can be extended to a solution $R = \mathrm{diag}(-1, R^0, 1)$ of the same equation on $\mathfrak{g}$. Moreover, if R^0 has no eigenvalues ± 1, then $\mathfrak{g}^0$ is solvable and R^0 is the Cayley transform of an automorphism A of G^0 without fixed points: $R^0 = (A + 1)(A - 1)^{-1}$. Conversely, any R-matrix R on a metrical Lie algebra $\mathfrak{g}$ defines some Gauss decomposition.

In 2.15 we give some simple constructions of Gauss decompositions of a metrical Lie algebra and its associated R-matrix. Remark that the problem of describing all bialgebra structures on a given semisimple Lie algebra $\mathfrak{g}_+$ (or the equivalent problem of determining all Manin decompositions $\mathfrak{g} = \mathfrak{g}_+ \oplus \mathfrak{g}_-$ of metrical Lie algebras $\mathfrak{g}$ with given $\mathfrak{g}_+$) is solved only for a simple Lie algebra $\mathfrak{g}_+$, [6,9]. The construction of Weinstein of a bialgebra structure on a compact semisimple Lie algebra shows that in general the isotropic subalgebras $\mathfrak{g}_+$, $\mathfrak{g}_-$ of a Gauss decomposition of a semisimple Lie algebra $\mathfrak{g}$ are not necessarily solvable. However, this is true if the metric g coincides with the Killing form of $\mathfrak{g}$, see [9].

The second part of the paper is devoted to explicit description of global versions of some objects which are studied in the first part. The basic object is the double Lie group G which corresponds to a metrical Lie algebra $\mathfrak{g}$ with a Manin decomposition $\mathfrak{g} = \mathfrak{g}_+ \oplus \mathfrak{g}_-$. We describe explicitly different natural Poisson and affine Poisson structures on a double group G and the dressing action of subgroups G_+, G_- associated with the isotropic subalgebras $\mathfrak{g}_+$, $\mathfrak{g}_-$.

2. Lie bialgebras, Manin triples, and Gauss decompositions

2.1. Lie bialgebras and Lie–Poisson groups. A *Lie bialgebra* [11] consists of a (finite-dimensional) Lie algebra $\mathfrak{g}$ with Lie bracket $b = [\ ,\ ] \in \wedge^2\mathfrak{g}^* \otimes \mathfrak{g}$ and an element $b' \in \wedge^2\mathfrak{g} \otimes \mathfrak{g}^*$ such that the following two properties hold:

(1) b' is a 1-cocycle $\mathfrak{g} \to \wedge^2\mathfrak{g}$: $\partial_b b' = 0$ where $(\partial_b b')(X, Y) = -b'([X, Y]) + \mathrm{ad}_X(b'(Y)) - \mathrm{ad}_Y(b'(X))$. To put this into perspective, note that this is equivalent to the fact that $X \mapsto (X, b'(X))$ is a homomorphism of Lie algebras from $\mathfrak{g}$ into the semidirect product $\mathfrak{g} \ltimes \wedge^2 \mathfrak{g}$ with the Lie bracket $[(X, U), (Y, V)] = ([X, Y], \mathrm{ad}_X V - \mathrm{ad}_Y U)$.

(2) b' is a Lie bracket on $\mathfrak{g}^*$.

In [17] a graded Lie bracket on $\wedge(\mathfrak{g} \times \mathfrak{g}^*)$ is constructed which recognizes Lie bialgebras, their representations, and gives the associated notion of Chevalley cohomology.

2.2. Exact Lie bialgebras and Yang–Baxter equations. A Lie bialgebra $(\mathfrak{g}, b, b')$ is called *exact* if the 1-cocycle b' is a coboundary: $b' = \partial_b C$ for $C \in \wedge^2\mathfrak{g}$, i.e., $b'(X) = \mathrm{ad}_X C$. A bivector $C \in \wedge^2\mathfrak{g}$ defines a Lie bialgebra structure $b' = \partial_b C$ on $\mathfrak{g}$ if and only if the Schouten bracket (see 3.4) is $\mathrm{ad}(\mathfrak{g})$-*invariant*:

$$\text{(mYBE)} \qquad [C, C] \in (\wedge^3\mathfrak{g})^{\mathfrak{g}}.$$

This condition is called the *modified Yang–Baxter Equation*. In particular any Poisson bivector $C \in \wedge^2\mathfrak{g}$ satisfying

$$\text{(YBE)} \qquad [C, C] = 0$$

defines a bialgebra structure $b' = \partial_b C$ in $\mathfrak{g}$. This equation is called the *Yang–Baxter Equation.*

If $\mathfrak{g}$ is semisimple then by the Whitehead lemma $H^1(\mathfrak{g}, \wedge^2\mathfrak{g}) = 0$, so any cocycle b' is a coboundary, and the classification of all bialgebra structures on $\mathfrak{g}$ reduces to the description of all bivectors $C \in \wedge^2\mathfrak{g}$ which satisfy (mYBE). If moreover the Lie algebra $\mathfrak{g}$ is simple then the space $(\wedge^3\mathfrak{g})^{\mathfrak{g}}$ is one-dimensional, generated by the 3-vector $B^g \in \wedge^3\mathfrak{g}$ given by $B^g(\alpha, \beta, \gamma) := g([g^{-1}\alpha, g^{-1}\beta], g^{-1}\gamma)$, where g denotes the Cartan–Killing form. So for simple $\mathfrak{g}$ the modified Yang–Baxter Equation (mYBE) can be written, using the Schouten bracket, as

$$[C, C] = cB^g.$$

All solutions of this equation for $c \neq 0$ for complex simple $\mathfrak{g}$ were described in [6,9].

2.3. Manin decompositions. Let $(\mathfrak{g}, b)$ be a Lie algebra and let b' be a Lie bracket on the dual space $\mathfrak{g}^*$. Let us define a skew symmetric bracket $[\ ,\ ]$ on the vector space $\mathfrak{l} := \mathfrak{g} \oplus \mathfrak{g}^*$ by

$$\begin{aligned}[(X, \alpha), (Y, \beta)] := (&b(X, Y) + \mathrm{ad}^*_{b'}(\alpha)Y - \mathrm{ad}^*_{b'}(\beta)X, \\ &b'(\alpha, \beta) + \mathrm{ad}^*_b(X)\beta - \mathrm{ad}^*_b(Y)\alpha),\end{aligned}$$

where $\mathrm{ad}_b(X)Y = b(X, Y)$, $\mathrm{ad}^*_b(X) = \mathrm{ad}_b(-X)^* \in \mathrm{End}(\mathfrak{g}^*)$, and similarly for b'. The adjoint operator $\mathrm{ad}(X, \alpha) \in \mathrm{End}(\mathfrak{l})$ is skew symmetric with respect to the natural pseudo-Euclidean inner product g on $\mathfrak{l}$ which is given by $g((X, \alpha), (Y, \beta)) = \langle\alpha, Y\rangle + \langle\beta, X\rangle$, and the skew symmetric bracket is uniquely determined by this property. The skew symmetric bracket $[\ ,\ ]$ on $\mathfrak{l}$ satisfies the Jacobi identity if and only if $b' : \mathfrak{g} \to \wedge^2\mathfrak{g}$ is a 1-cocycle with respect to b: $\partial_b b' = 0$; or equivalently if and only if $b : \mathfrak{g}^* \to \wedge^2\mathfrak{g}^*$ is a 1-cocycle with respect to b': $\partial_{b'} b = 0$.

Following Astrakhantsev [4] we will call *metrical Lie algebra* a Lie algebra $\mathfrak{l}$ together with an ad-invariant inner product g: $g([X, Y], Z) = g(X, [Y, Z])$.

A decomposition of a metrical Lie algebra $(\mathfrak{l}, g)$ as a direct sum $\mathfrak{l} = \mathfrak{g}_+ \oplus \mathfrak{g}_-$ of two g-isotropic Lie subalgebras $\mathfrak{g}_+$ and $\mathfrak{g}_-$ is called a *Manin decomposition.*

A triple of Lie algebras $(\mathfrak{g}, \mathfrak{g}_+, \mathfrak{g}_-)$ together with a duality pairing between $\mathfrak{g}_+$ and $\mathfrak{g}_-$ is called a *Manin triple* if $\mathfrak{g} = \mathfrak{g}_+ \oplus \mathfrak{g}_-$, $\mathfrak{g}_+$ and $\mathfrak{g}_-$ are Lie algebras of $\mathfrak{g}$, and the duality pairing induces an *ad*-invariant inner product on $\mathfrak{g}$ for which $\mathfrak{g}_+$ and $\mathfrak{g}_-$ are isotropic.

Theorem [10]. *There exist a natural bijective correspondence between Lie bialgebras* $(\mathfrak{g}, b, b')$ *and metrical Lie algebras* $(\mathfrak{l}, g)$ *with Manin decomposition* $\mathfrak{l} = \mathfrak{g} \oplus \mathfrak{g}^*$.

The Lie algebra $\mathfrak{l} = \mathfrak{g} \oplus \mathfrak{g}^*$ associated to the Lie bialgebra $(\mathfrak{g}, b, b')$ is called the *Manin double.*

2.3. Examples of metrical Lie algebras. Any commutative Lie algebra has the structure of a metrical Lie algebra, with respect to any inner product. Any semisimple Lie algebra is metrical, the metric is given by the Cartan–Killing form.

Let $\mathfrak{g}$ be a Lie algebra. Let us denote by $T^*\mathfrak{g} = \mathfrak{g} \ltimes \mathfrak{g}^*$ the semidirect product of the Lie algebra $\mathfrak{g}$ with the abelian ideal $\mathfrak{g}^*$, where $\mathfrak{g}$ acts on $\mathfrak{g}^*$ by the the coadjoint action. This is the Lie algebra of the cotangent group T^*G of a Lie group G with Lie algebra $\mathfrak{g}$. The natural pairing between $\mathfrak{g}$ and the dual $\mathfrak{g}^*$ defines an ad-invariant inner product g on $T^*\mathfrak{g}$ for which the subalgebras $\mathfrak{g}$ and $\mathfrak{g}^*$ are isotropic, by definition of the coadjoint action. Hence $T^*\mathfrak{g} = \mathfrak{g} \oplus \mathfrak{g}^*$ is a Manin decomposition of the metrical Lie algebra $T^*\mathfrak{g}$. It describes the Lie bialgebra structure $b' = 0$ on $\mathfrak{g}$.

2.5. We will now describe the *double extension* of a metrical Lie algebra according to Kac [14, 2.10] and Medina and Revoy [25]: Let $(\mathfrak{g}, g)$ be a metrical Lie algebra and let $\mathfrak{d}$ be a Lie algebra together with a representation $\rho : \mathfrak{d} \to \mathrm{Der}_{\mathrm{skew}}(\mathfrak{g}, g)$ by skew symmetric derivations on $\mathfrak{g}$. We then put

$$\begin{aligned}
&\mathfrak{g}_{\mathfrak{d}} := \mathfrak{d} \oplus \mathfrak{g} \oplus \mathfrak{d}^*,\\
&[D_1 + X_1 + \alpha_1, D_2 + X_2 + \alpha_2]\\
&\quad = [D_1, D_2]_{\mathfrak{d}} + [X_1, X_2]_{\mathfrak{g}} + \rho(D_1)(X_2) - \rho(D_2)(X_1)\\
&\qquad + c(X_1, X_2) + \mathrm{ad}^*_{\mathfrak{d}}(D_1)(\alpha_2) - \mathrm{ad}^*_{\mathfrak{d}}(D_2)(\alpha_1),\\
&g_{\mathfrak{g}_{\mathfrak{d}}}(D_1 + X_1 + \alpha_1, D_2 + X_2 + \alpha_2)\\
&\quad = g(X_1, X_2) + \langle \alpha_1, D_2 \rangle + \langle \alpha_2, D_1 \rangle,
\end{aligned}$$

where the central cocycle $c : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{d}^*$ is given by $\langle D, c(X, Y) \rangle = g(\rho(D)(X), Y)$ for $D \in \mathfrak{d}$. Then $\mathfrak{g}_{\mathfrak{d}}$ is again a metrical Lie algebra. Note that the metrical Lie subalgebra $\mathfrak{d} \oplus \mathfrak{d}^*$ is isomorphic to the cotangent Lie algebra $T^*\mathfrak{d}$ and that we may view $\mathfrak{g}_{\mathfrak{d}}$ as the semidirect product $\mathfrak{g}_{\mathfrak{d}} = \mathfrak{d} \ltimes \mathfrak{h}$, where $\mathfrak{h}$ is the central extension

$$0 \to \mathfrak{d}^* \to \mathfrak{h} \to \mathfrak{g} \to 0$$

described by the cocycle c and where $\mathfrak{d}$ acts on $\mathfrak{h}$ by $(\rho, \mathrm{ad}^*_{\mathfrak{d}})$.

The orthogonal direct sum of two metrical Lie algebras is again a metrical Lie algebra. In particular the orthogonal direct sum of a metrical Lie algebra $\mathfrak{g}$ with a one-dimensional abelian metrical Lie algebra is called the *trivial extension* of $\mathfrak{g}$.

Theorem (Kac [14, 2.11]; Revoy, Medina [25]). *Any solvable metrical Lie algebra can be obtained from a commutative metrical Lie algebra by an appropriate sequence of double extensions and trivial extensions.*

2.6. The following result gives an analogon of the Levi-Maltsev decomposition for a metrical Lie algebra.

Theorem (Astrachantsev [4]). *Any metrical Lie algebra $\mathfrak{g}$ is an orthogonal direct sum*

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{r} = \mathfrak{s}_n \oplus T^*\mathfrak{s}_i \oplus \mathfrak{r}$$

consisting of a subalgebra $\mathfrak{h}$ with commutative radical and a solvable ideal $\mathfrak{r}$. Moreover, $\mathfrak{h}$ is an orthogonal direct sum of a maximal g-non-degenerate semisimple Lie subalgebra $\mathfrak{s}_n$ and the cotangent algebra $T^*\mathfrak{s}_i$ of a maximal g-isotropic semisimple Lie subalgebra $\mathfrak{s}_i$ of $\mathfrak{g}$.

2.7. Metrical extensions. Bordemann [8] gave the following construction of a metrical Lie algebras.

Let $\mathfrak{a}$ be a Lie algebra and let $w : \mathfrak{a} \wedge \mathfrak{a} \to \mathfrak{a}^*$ be a 2-cocyle with values in the $\mathfrak{a}$-module $\mathfrak{a}^*$. Then the Lie algebra extension

$$0 \to \mathfrak{a}^* \to \mathfrak{g}_w \to \mathfrak{a} \to 0$$

described by w, i.e. the Lie algebra $\mathfrak{g}_w := \mathfrak{a} \oplus \mathfrak{a}^*$ with bracket

$$[(a,\alpha),(b,\beta)]_{\mathfrak{g}_w} := ([a,b]_{\mathfrak{g}}, w(a,b) + \mathrm{ad}^*(a)\beta - \mathrm{ad}^*(b)\alpha)$$

is a metrical Lie algebra with metric

$$g((a,\alpha),(b,\beta)) := \langle \alpha, b\rangle + \langle \beta, a\rangle$$

if and only if w has the following property

$$\langle a, w(b,c)\rangle = \langle b, w(c,a)\rangle \quad \text{for } a,b,c \in \mathfrak{a}.$$

If $w = 0$ then this is exactly the metrical Lie algebra $T^*\mathfrak{a}$ – thus Bordemann called this construction the T^*-extension.

Theorem [8]. *Any $2n$-dimensional complex solvable metrical Lie algebra $\mathfrak{g}$ is a metrical extension of some n-dimensional Lie algebra $\mathfrak{a}$. Moreover any isotropic ideal of $\mathfrak{g}$ is contained in an n-dimensional isotropic commutative ideal of $\mathfrak{g}$.*

2.8. The Yang–Baxter equations on metrical Lie algebras.

In the case of a metrical Lie algebra $(\mathfrak{g}, g)$ we can pull down one index of bivector $C \in \wedge^2 g$ and we can reformulate the (modified) Yang–Baxter equation in terms of the operator $R = C \circ g : \mathfrak{g} \to \mathfrak{g}^* \to \mathfrak{g}$.

First let $(\mathfrak{g}, b = [\ ,\ ])$ be a Lie algebra. For any $R \in \mathrm{End}(\mathfrak{g})$ we define two elements $b_R, B_R \in \mathfrak{g} \otimes \wedge^2\mathfrak{g}^*$ by

$$b_R(X,Y) = [X,Y]_R := [RX,Y] + [X,RY],$$
$$B_R(X,Y) := [RX,RY] - R[X,Y]_R = [RX,RY] - R[RX,Y] - R[X,RY].$$

Note that B_R is related to the Frölicher–Nijenhuis-like bracket $[R,R]$ by

$$\begin{aligned}\tfrac{1}{2}[R,R](X,Y) &= [RX,RY] - R([RX,Y] + [X,RY]) + R^2[X,Y]\\ &= B_R(X,Y) + R^2[X,Y].\end{aligned}$$

Proposition. *Let $(\mathfrak{g}, g)$ be a metrical Lie algebra, let $C \in \wedge^2\mathfrak{g}$ and let $R = C \circ g : \mathfrak{g} \to \mathfrak{g}^* \to \mathfrak{g}$ be the corresponding operator. Then we have:*

(1) *Via the isomorphism $g^{-1} : \mathfrak{g}^* \to \mathfrak{g}$ the bracket $b' = \partial_b C \in \mathfrak{g}^* \otimes \wedge^2\mathfrak{g}$ on $\mathfrak{g}^*$ corresponds to the bracket b_R on $\mathfrak{g}$:*

$$g^{-1}(b'(\alpha,\beta)) = b_R(g^{-1}\alpha, g^{-1}\beta) = [g^{-1}\alpha, g^{-1}\beta]_R, \quad \text{for } \alpha,\beta \in \mathfrak{g}^*.$$

(2) *Under the embedding* $\wedge^3\mathfrak{g} \to \mathfrak{g} \otimes \wedge^2\mathfrak{g}^*$ *induced by* g, *the Schouten bracket* $[C, C] \in \wedge^3\mathfrak{g}$ *corresponds to the element* $2B_R \in \mathfrak{g} \otimes \wedge^2\mathfrak{g}^*$.

Proof. Let $X, Y, Z \in \mathfrak{g}$ and $\alpha = gX, \beta = gY \in \mathfrak{g}^*$. Note that $g(RX, Y) = g(X, -RY)$. Then

$$\begin{aligned}
\langle Z, b'(\alpha, \beta)\rangle &= \langle \mathrm{ad}_Z C, \alpha \wedge \beta\rangle = \langle C, (\mathrm{ad}_Z)^*\alpha \wedge \beta + \alpha \wedge (\mathrm{ad}_Z)^*\beta\rangle \\
&= \langle C, (\mathrm{ad}_Z)^* gX \wedge gY + gX \wedge (\mathrm{ad}_Z)^* gY\rangle \\
&= \langle C, -g\mathrm{ad}_Z X \wedge gY - gX \wedge g\mathrm{ad}_Z Y\rangle \\
&= -\langle Cg\mathrm{ad}_Z X, gY\rangle - \langle CgX, g\mathrm{ad}_Z Y\rangle \\
&= -g(R[Z, X], Y) - g(RX, [Z, Y]) \\
&= g(Z, [X, RY]) + g([RX, Y], Z) \\
&= \langle Z, g[X, Y]_R\rangle.
\end{aligned}$$

For proving the second assertion we may assume without loss that $C \in \wedge^2\mathfrak{g}$ is decomposable, $C = X \wedge Y$, since both sides are quadratic. Then we have:

$$\begin{aligned}
R(Z) &= (C \circ g)(Z) = ((X \wedge Y) \circ g)(Z) = g(Y, Z)X - g(X, Z)Y, \\
B_R(U, V) &= [RU, RV] - R[RU, V] - R[U, RV] \\
&= [g(Y, U)X - g(X, U)Y, g(Y, V)X - g(X, V)Y] \\
&\quad - g(Y, [g(Y, U)X - g(X, U)Y, V])X \\
&\quad + g(X, [g(Y, U)X - g(X, U)Y, V])Y \\
&\quad - g(Y, [U, g(Y, V)X - g(X, V)Y])X \\
&\quad + g(X, [U, g(Y, V)X - g(X, V)Y])Y \\
&= -g(Y, U)g(X, V)[X, Y] - g(X, U)g(Y, V)[Y, X] \\
&\quad - g(Y, U)g([Y, X], V)X - g(X, U)g([X, Y], V)Y \\
&\quad + g(Y, V)g([Y, X], U)X + g(X, V)g([X, Y], U)Y.
\end{aligned}$$

On the other hand we have for the Schouten bracket

$$\begin{aligned}
&[C, C] = [X \wedge Y, X \wedge Y] = 2[X, Y] \wedge X \wedge Y, \\
&\tfrac{1}{2}\langle [C, C], \alpha \wedge gU \wedge gV\rangle \\
&\quad = \langle [X, Y] \wedge X \wedge Y, \alpha \wedge gU \wedge gV\rangle \\
&\quad = \det\begin{pmatrix} \langle [X, Y], \alpha\rangle & \langle X, \alpha\rangle & \langle Y, \alpha\rangle \\ g([X, Y], U) & g(X, U) & g(Y, U) \\ g([X, Y], V) & g(X, V) & g(Y, V) \end{pmatrix} \\
&\quad = \langle B_R(U, V), \alpha\rangle,
\end{aligned}$$

from the computation above. □

Remarks. We may extend $R \mapsto B_R$ to a bracket in $\wedge\mathfrak{g}^* \otimes \mathfrak{g}$ as follows. On decomposable tensors this bracket is given by

$$[\varphi \otimes X, \psi \otimes Y] = \varphi \wedge \psi \otimes [X, Y] + \varphi \wedge \mathrm{ad}_X^* \psi \otimes Y \\ - \mathrm{ad}_Y^* \varphi \wedge \psi \otimes X,$$

and it defines a $\mathbb{Z}$-graded Lie bracket on $\wedge^*\mathfrak{g}^* \otimes \mathfrak{g}$. If $\mathfrak{g}$ acts by derivations on a graded commutative algebra $A = \bigotimes_{i=0}^{\infty} A_i$, the same formulae define a graded Lie bracket on $A \otimes \mathfrak{g}$.

Moreover we have $B_R = \frac{1}{2}[R, R]^B$, and by the graded Jacobi identity we get the analogon of the Bianchi identity $[R, B_R]^B = 0$.

The invariant inner product $g : \mathfrak{g} \to \mathfrak{g}^*$ induces an embedding

$$\wedge^{*+1}\mathfrak{g} \to \wedge^*\mathfrak{g}^* \otimes \mathfrak{g},$$

which is a homomorphism from the Schouten bracket to the graded Lie bracket $[\ ,\ ]^B$. This follows from the polarization of (2) in the proposition above (note that the brackets in degree 1 are symmetric), since $\mathfrak{g}$ and $\wedge^2\mathfrak{g}$ generate the whole Schouten algebra.

On a manifold one may also consider the bracket $[\ ,\ ]^B$ but it maps tensor fields to differential operators.

There is a homomorphism of graded Lie algebras

$$(\wedge^*\mathfrak{g}^* \otimes \mathfrak{g}, [\ ,\ ]^B) \to (\Omega^*(\mathfrak{g}, \mathfrak{g}), [\ ,\ ]^{FN}),$$
$$\alpha_1 \wedge \cdots \wedge \alpha_p \otimes X \mapsto d\alpha_1 \wedge \cdots \wedge d\alpha_p \otimes \mathrm{ad}_{\mathfrak{g}}(X),$$

where $\Omega(\mathfrak{g}, \mathfrak{g}) \cong \Omega(\mathfrak{g}; T\mathfrak{g})$ is the graded Lie algebra of all tangent space valued differential forms on $\mathfrak{g}$ with the Frölicher–Nijenhuis bracket. The kernel of this homomorphism consists of $\wedge^*\mathfrak{g}^* \otimes Z(\mathfrak{g})$ where $Z(\mathfrak{g})$ is the center of $\mathfrak{g}$. All these follow from the well-known formula for the Frölicher–Nijenhuis bracket (see e.g. [16, 8.7])

$$[\varphi \otimes \xi, \psi \otimes \eta] = \varphi \wedge \psi \otimes [\xi, \eta] + \varphi \wedge \mathcal{L}_\xi \psi \otimes \eta - \mathcal{L}_\eta \varphi \wedge \psi \otimes \xi \\ + (-1)^{\deg \varphi}(d\varphi \wedge i_\xi \psi \otimes \eta + i_\eta \varphi \wedge d\psi \otimes \xi),$$

where $\varphi, \psi \in \Omega(\mathfrak{g})$ are differential forms and where $\xi, \eta \in \mathfrak{X}(\mathfrak{g})$ are vector fields.

2.9. Corollary (see [32]). *For $C \in \wedge^2\mathfrak{g}$ and $R = C \circ g : \mathfrak{g} \to \mathfrak{g}$ the following conditions are equivalent.*

(1) *$b' = \partial C$ is a Lie bracket in $\mathfrak{g}^*$, hence $(\mathfrak{g}, b, b')$ is a Lie bialgebra.*
(2) *b_R is a Lie bracket in $\mathfrak{g}$.*
(3) *The Schouten bracket $[C, C] \in \wedge^3\mathfrak{g}$ is $\mathrm{ad}_{\mathfrak{g}}$-invariant.*
(4) *$B_R \in (\mathfrak{g} \otimes \wedge^2\mathfrak{g}^*)$ is $\mathfrak{g}$-invariant.*
(5) *For all $X, Y, Z \in \mathfrak{g}$ we have*

$$[X, B_R(Y, Z)] + [Y, B_R(Z, X)] + [Z, B_R(X, Y)] = 0.$$

Proof. It remains to show that (4) is equivalent to (5). This follows from the identity

$$g((\operatorname{ad}(U)B_R)(Y,Z),X)$$
$$= -g([X,B_R(Y,Z)]+[Y,B_R(Z,X)]+[Z,B_R(X,Y)],U),$$

which holds for all $X, Y, Z, U \in \mathfrak{g}$. □

The following simpler equations obviously imply Eq. (5):

(I-mYBE)	$B_R + I \circ b = 0$	or $B_R(X,Y) + I[X,Y] = 0$,
(c-mYBE)	$B_R + cb = 0$	or $B_R(X,Y) + c[X,Y] = 0$,
(YBE)	$[C,C] = 0$	or $B_R = 0$,

where $I \in \operatorname{End}(\mathfrak{g})^{\mathfrak{g}}$ is an $\operatorname{ad}_{\mathfrak{g}}$-invariant operator on $\mathfrak{g}$, and where c is a constant in $\mathbb{K}$. If $\mathbb{K} = \mathbb{C}$ (or $\mathbb{K} = \mathbb{R}$) without loss we may assume that $c = 1$ (or $c = \pm 1$).

In [9, 3.2], it was shown that any structure of a bialgebra on a semisimple Lie algebra comes from a solution of (I-mYBE) for some $I \in \operatorname{End}(\mathfrak{g})^{\mathfrak{g}}$; and for a simple Lie algebra from a solution of (c-mYBE).

It is also interesting to construct non-skew symmetric solutions of all these equations. Some class of solutions on a simple complex Lie algebra was constructed in [31].

Note that for an $\operatorname{ad}_{\mathfrak{g}}$-invariant operator $I \in \operatorname{End}(\mathfrak{g})^{\mathfrak{g}}$ we have $B_I = I^2 \circ b$ since $I[X,Y] = [IX,Y] = [X,IY]$. So any skew symmetric ad-invariant operator I gives a solution of the (mYBE). Non-constant operators of this kind exist on semisimple Lie algebras $\mathfrak{g}$ if and only if $\mathfrak{g}$ has isomorphic simple summands: For example, if $\mathfrak{g} = l\mathfrak{g}_1 = \mathfrak{g}_1 \oplus \ldots \oplus \mathfrak{g}_1 = \mathfrak{g}_1 \otimes \mathbb{K}^l$ then $\operatorname{End}(\mathfrak{g})^{\mathfrak{g}} = 1 \otimes \operatorname{End}(\mathbb{K}^l)$, and any skew symmetric matrix $A \in \operatorname{End}(\mathbb{K}^l)$ gives a solution $I = 1 \otimes A$ of (mYBE).

To distinguish equations for $C \in \wedge^2\mathfrak{g}$ and for $R = C \circ g$ the equation (1-mYBE) for R will be called the *R-matrix equation*, and solutions will be called *R-matrices*.

2.10. Let $(\mathfrak{g}, b, g)$ be a metrical Lie-algebra and let $R \in \operatorname{End}(\mathfrak{g})$ be a skew symmetric endomorphism.

Lemma [9,32]. *The following conditions are equivalent:*

(1) *The endomorphism R satisfies the R-matrix equation $B_R + b = 0$.*

(2) *The endomorphisms $R_\pm := R \pm 1$ satisfy*

$$R_+[R_-X, R_-Y] = R_-[R_+X, R_+Y] \quad \text{for } X, Y \in \mathfrak{g}.$$

(3) *For all $\lambda, \mu \in \mathbb{C}$ and $X, Y \in \mathfrak{g}$ we have*

$$(\lambda+\mu)R[X,Y] = (1+\lambda\mu)[X,Y] + [(R-\lambda)X,(R-\mu)Y]$$
$$-(R-\lambda)[X,(R-\mu)Y] - (R-\mu)[(R-\lambda)X,Y].$$

(4) *The bracket $b_R(X,Y) = [X,Y]_R = [RX,Y] + [X,RY]$ is a Lie bracket and moreover both $R_\pm : (\mathfrak{g}, b_R) \to (\mathfrak{g}, b)$ are homomorphisms.*

2.11. For an endomorphism $R : \mathfrak{g} \to \mathfrak{g}$ and $\lambda \in \mathbb{C}$ the space

$$\mathfrak{g}_\lambda = \ker(R-\lambda)^N \quad \text{for large } N$$

is called *weight space* if it is not 0, and λ is called weight of R. We have the following decomposition of $\mathfrak{g}$ into a direct sum of all weight spaces:

$$\mathfrak{g} = \bigoplus_{\lambda \in W} \mathfrak{g}_\lambda,$$

where W is the set of all weights.

For $\lambda, \mu \in \mathbb{C}$ with $\lambda + \mu \neq 0$ we put

$$(1) \qquad \lambda \circ \mu := \frac{1 + \lambda\mu}{\lambda + \mu}. \tag{2.1}$$

Note that $(\pm 1) \circ \mu = \pm 1$.

Lemma [9]. *Let R be an R-matrix on a metrical Lie algebra $(\mathfrak{g}, g)$. Then we have:*
(1) *For weights λ, μ with $\lambda + \mu \neq 0$ we have*

$$[\mathfrak{g}_\lambda, \mathfrak{g}_\mu] \subseteq \mathfrak{g}_{\lambda \circ \mu} \quad \text{and} \quad g(\mathfrak{g}_\lambda, \mathfrak{g}_\mu) = 0.$$

(2) *For $\lambda \neq \pm 1$ we have $[\mathfrak{g}_\lambda, \mathfrak{g}_{-\lambda}] = 0$.*
(3) *The spaces $\mathfrak{g}_{\pm 1}$ are Lie subalgebras of $\mathfrak{g}$, and $[\mathfrak{g}_\lambda, \mathfrak{g}_{\pm 1}] \subseteq \mathfrak{g}_{\pm 1}$ for $\lambda \neq \pm 1$.*

2.12. R-matrices and associated Gauss decompositions. We will discuss the relations between R-matrices on a metrical Lie algebra and its Gauss decompositions.

Definition. A *(generalized) Gauss decomposition* of a metrical Lie algebra $(\mathfrak{g}, g)$ is a decomposition of $\mathfrak{g}$

$$\mathfrak{g} = \mathfrak{g}_+ \oplus \mathfrak{g}^0 \oplus \mathfrak{g}_-$$

into a sum of subalgebras, where the inner product g is non-degenerate on $\mathfrak{g}^0$, and where $\mathfrak{g}_+$ and $\mathfrak{g}_-$ are isotropic subalgebras which are orthogonal to $\mathfrak{g}^0$.

Note that a Manin decomposition is the special case of a Gauss decomposition with $\mathfrak{g}^0 = 0$.

Proposition. *An R-matrix R on a metrical Lie algebra $(\mathfrak{g}, g)$ defines a Gauss decomposition*

$$\mathfrak{g} = \mathfrak{g}_- \oplus \mathfrak{g}^0 \oplus \mathfrak{g}_+,$$

where $\mathfrak{g}_\pm$ are the weight spaces $\mathfrak{g}_{\pm 1}$ of R, and where

$$\mathfrak{g}^0 = \bigoplus_{\lambda \neq \pm 1} \mathfrak{g}_\lambda$$

is a solvable Lie subalgebra which admits an g-orthogonal automorphism $A = ((R + 1)|\mathfrak{g}^0) \circ ((R - 1)|\mathfrak{g}^0)^{-1}$ without fixed point (so $AX = X$ implies $X = 0$).

Conversely, let $\mathfrak{g} = \mathfrak{g}_- \oplus \mathfrak{g}^0 \oplus \mathfrak{g}_+$ *be a Gauss decomposition of a metrical Lie algebra* $(\mathfrak{g}, g)$*, where* g^0 *admits an orthogonal automorphism* A *without fixed points. Put* $R_0 = (A+1) \circ (A-1)^{-1}$*. Then*

$$R = \operatorname{diag}(-1, R_0, 1) : \mathfrak{g} \to \mathfrak{g}$$

is an R*-matrix.*

More generally, any R*-matrix* R' *on* $\mathfrak{g}$ *which induces this Gauss decomposition has the form*

$$R' = \operatorname{diag}(-1 + N_-, R_0, 1 + N_+) : \mathfrak{g} \to \mathfrak{g},$$

where $N_\pm : \mathfrak{g}_\pm \to \mathfrak{g}_\pm$ *are suitable nilpotent endomorphisms.*

Remark that, in fact, the R-matrix equation specifies the form of $N_\pm$. For example, denote by $\mathfrak{g}^i_\pm = \ker(N_\pm)^i \subseteq \mathfrak{g}_\pm$. Then

$$\mathfrak{g}_\pm = \mathfrak{g}^k_\pm \supset \mathfrak{g}^{k-1}_\pm \supset \cdots \supset \mathfrak{g}^1_\pm \supset \mathfrak{g}^0_\pm = 0$$

is a chain of ideals: $[\mathfrak{g}^i_\pm, \mathfrak{g}_\pm] \subset \mathfrak{g}^i_\pm$.

Proof. The first statement follows immediately from Lemma 2.11. The operators $R_\pm|\mathfrak{g}^0$ are invertible. Note that by putting $X = (R-1)^{-1}u$ and $Y = (R-1)^{-1}v$ for $u, v \in \mathfrak{g}^0$ the equation in Lemma 2.10(2) becomes $(R+1)(R-1)^{-1}[u, v] = [(R+1)(R-1)^{-1}u, (R+1)(R-1)^{-1}v]$. This shows that $A = (R+1)(R-1)^{-1}$ is an automorphism of $\mathfrak{g}^0$. It has no fixed point. It is easily seen that A is orthogonal if and only if $R|g^0$ is skew symmetric.

We now use the fact that a Lie algebra which admits an automorphism without fixed point is solvable, see [37].

For the converse, since all arguments above were equivalencies, we see that $R_0 = (A+1)(A-1)^{-1}$ is a (skew symmetric) R-matrix on $\mathfrak{g}^0$. Using Lemma 2.10(2) again it follows by checking cases $X, Y \in \mathfrak{g}_-, \mathfrak{g}_+, \mathfrak{g}_0$ that $R = \operatorname{diag}(-1, R_0, 1)$ is an R-matrix.

The last statement is obvious. □

2.13. Corollary. *Any semisimple* R*-matrix* R *on a metrical Lie algebra* $(\mathfrak{g}, g)$ *can be written as*

$$R = \operatorname{diag}(-1, R_0, 1)$$

with respect to an appropriate Gauss decomposition $\mathfrak{g} = \mathfrak{g}_- \oplus \mathfrak{g}^0 \oplus \mathfrak{g}_+$*, where* $R_0 = (A+1)(A-1)^{-1}$ *for a semisimple orthogonal automorphism* A *of* $\mathfrak{g}_0$ *without fixed point.*

2.14. Corollary. *Any* R*-matrix* R *on a metrical Lie algebra* $(\mathfrak{g}, g)$ *without eigenvalues* ± 1 *is of the form*

$$R = (A+1) \circ (A-1)^{-1},$$

where A *is an orthogonal automorphism of* $\mathfrak{g}$ *without fixed point.*

Note that non-orthogonal automorphisms A give non-skew symmetric solutions $R = (A+1)\circ(A-1)^{-1}$ of the R-matrix equation.

2.15. Construction of R-matrices via Gauss decompositions. Let $(\mathfrak{g}, g)$ be a metrical Lie algebra. Choose a skew-symmetric derivation D of $\mathfrak{g}$ (for example an inner derivation $\mathrm{ad}(X_0)$ for $X_0 \in \mathfrak{g}$). It defines a decomposition

$$\mathfrak{g} = \mathfrak{g}_- \oplus \mathfrak{g}^0 \oplus g_+, \quad \text{where } \mathfrak{g}^0 = \mathfrak{g}_0$$

and

$$\mathfrak{g}_+ = \bigoplus_{\substack{\Re(\lambda)>0 \text{ or} \\ \Re(\lambda)=0,\Im(\lambda)>0}} \mathfrak{g}_\lambda, \qquad \mathfrak{g}_- = \bigoplus_{\substack{\Re(\lambda)>0 \text{ or} \\ \Re(\lambda)=0,\Im(\lambda)>0}} \mathfrak{g}_{-\lambda}.$$

Lemma. *For a complex Lie algebra this decomposition associated to a skew symmetric derivation D is a Gauss decomposition.*

Proof. $g((D-\mu)^l X, Y) = g(X, (-D-\mu)^l Y)$. □

We can iterate this construction if there exists non-nilpotent skew symmetric derivations of $\mathfrak{g}_0$, in particular if $\mathfrak{g}_0$ is not nilpotent. Hence we have:

2.16. Corollary. *Let D be a skew symmetric derivation on $(\mathfrak{g}, g)$.*
The decomposition associated to D is trivial, $\mathfrak{g} = \mathfrak{g}^0$, if and only if D is nilpotent.
If 0 is not an eigenvalue of D then the associated decomposition is a Manin decomposition

$$\mathfrak{g} = \mathfrak{g}_+ \oplus \mathfrak{g}_-.$$

2.17. Remark. In the special case when the subalgebra $\mathfrak{g}^0$ of a Gauss decomposition is commutative, then for any skew symmetric endomorphism $R_0 : \mathfrak{g}^0 \to \mathfrak{g}^0$ the operator

$$R = \mathrm{diag}(-1, R_0, 1)$$

is an R-matrix. It is known, [12] or [24, 9.3.10], that the connected component of the stabilizer of a regular point in the coadjoint representation of any connected Lie group is commutative. For a metrical Lie algebra the adjoint representation is isomorphic to the coadjoint one. Hence the Gauss decomposition associated to an inner derivation $\mathrm{ad}(X)$ of a regular semisimple element $X \in \mathfrak{g}$ has $\mathfrak{g}^0$ commutative.

2.18. Construction of R-matrices without eigenvalues ± 1. Let $\mathfrak{g}$ be a (nilpotent) Lie algebra which admits a derivation with positive eigenvalues. For example, let $\mathfrak{g} = \bigoplus_{i>0} \mathfrak{g}_i$ be a positively graded Lie algebra and let $D|\mathfrak{g}_i = i\mathrm{Id}$. Denote by $T^*\mathfrak{g} = \mathfrak{g} \rhd\!< \mathfrak{g}^*$ the semidirect sum of $\mathfrak{g}$ and the commutative ideal $\mathfrak{g}^*$ with the coadjoint action on $\mathfrak{g}^*$. The natural pairing $\mathfrak{g} \times \mathfrak{g}^* \to \mathbb{C}$ defines an $\mathrm{ad}_{T^*\mathfrak{g}}$-invariant metric g on $\mathfrak{g}$. The derivation D can naturally be extended to a g-skew symmetric derivation D on $T^*\mathfrak{g}$ without eigenvalue

0. Then $A_t := \exp(tD)$ is a g-orthogonal automorphism of $(T^*\mathfrak{g}, g)$ without fixed point. Hence

$$R = (A_t + 1)(A_t - 1)^{-1}$$

is an R-matrix without eignevalues ± 1.

2.19. Proposition [32]. *Let $\mathfrak{g} = \mathfrak{g}_+ \oplus \mathfrak{g}_-$ be a Manin decomposition of a metrical Lie algebra $\mathfrak{g}$, and let $\mathrm{pr}_\pm : \mathfrak{g} \to \mathfrak{g}_\pm$ be the corresponding projections. Then $R = \mathrm{pr}_+ - \mathrm{pr}_-$ is a solution of (1-mYBE) $B_R + b = 0$.*

2.20. Proposition. *Let $\mathfrak{g} = \mathfrak{g}_+ \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_-$ be a Gauss-decomposition of a metrical Lie algebra $\mathfrak{g}$, and let $\mathrm{pr}_\pm : \mathfrak{g} \to \mathfrak{g}_\pm$ be the orthogonal projections. Then any solution R_0 of the (1-mYBE) on $\mathfrak{g}_0$ has an extension $R = c(\mathrm{pr}_+ \oplus R_0 \oplus (1-c)\mathrm{pr}_-)$ to a solution of the (1-mYBE) on $\mathfrak{g}$.*

This gives us an inductive procedure for the construction of solutions of the (mYBE).

2.21. Theorem. *Let $(\mathfrak{g}, g)$ be a metrical Lie algebra and let $R : \mathfrak{g} \to \mathfrak{g}$ be a solution of* 2.9, *(1-mYBE). Then the following Manin decompositions are isomorphic:*

(1) *The Manin double $\mathfrak{g} \oplus \mathfrak{g}^*$ associated to the bialgebra structure $b' = \partial_b(R \circ g^{-1})$ from* 2.3.

(2) *The direct sum $\mathfrak{g} \oplus \mathfrak{g} = \mathfrak{g}_{\mathrm{diag}} \oplus \mathfrak{g}_R$ with the metric $g_2((X, Y), (X, Y)) = g(X, X) - g(Y, Y)$ for $(X, Y) \in \mathfrak{g} \oplus \mathfrak{g}$, where $\mathfrak{g}_{diag} = \{(X, X): X \in \mathfrak{g}\}$ is isomorphic to $\mathfrak{g}$, and where the subalgebra $\mathfrak{g}_R = \{((R+1)X, (R-1)X): X \in \mathfrak{g}\}$ is isomorphic to the Lie algebra $(\mathfrak{g}, b_R)$ with bracket $b_R(X, Y) = [RX, Y] + [X, RY]$, which again is isomorphic to $(\mathfrak{g}^*, b')$, see* 2.8.

Proof. For an R-matrix R the mapping $(R+1, R-1) : (\mathfrak{g}, b_R) \to \mathfrak{g} \times \mathfrak{g}$ is a homomorphism of Lie algebras into the direct product by Lemma 2.10, which is injective. Also by Lemma 2.8 the mapping $g : (\mathfrak{g}, b_R) \to (\mathfrak{g}^*, b')$ is an isomorphism of Lie algebras. The direct sum Lie algebra $\mathfrak{g} \oplus \mathfrak{g}$ admits a decomposition into Lie subalgebras

$$\mathfrak{g} \oplus \mathfrak{g} = \{(X, X): X \in \mathfrak{g}\} \oplus \{((R+1)Y, (R-1)Y): Y \in \mathfrak{g}\},$$

where

$$(U, V) = (X, X) + ((R+1)Y, (R-1)Y),$$
$$2X = R(V - U) + V + U, \qquad 2Y = U - V,$$

which are isotropic:

$$g_2((R+1)Y, (R-1)Y) = g((R+1)Y, (R+1)Y) - g((R-1)Y, (R-1)Y) = 0$$

since R is skew symmetric for g. □

2.22. Remark. The construction of an R-matrix on a semisimple metrical Lie algebra $(\mathfrak{g}, g)$ reduces to the construction of a Manin decomposition $\mathfrak{g} \oplus \mathfrak{g} = \mathfrak{g}_- \oplus \mathfrak{g}_+$ of the metrical Lie algebra $(\mathfrak{g} \oplus \mathfrak{g}, g \oplus (-g))$ where $\mathfrak{g}_- = \mathfrak{g}_{\text{diag}}$ is the diagonal subalgebra.

3. Notation on Lie groups

3.1. Notation for Lie groups. Let G be a Lie group with Lie algebra $\mathfrak{g} = T_eG$, multiplication $\mu : G \times G \to G$, and for $g \in G$ let $\mu_g, \mu^g : G \to G$ denote the left and right translation, $\mu(g, h) = g \cdot h = \mu_g(h) = \mu^h(g)$.

Let $L, R : \mathfrak{g} \to \mathfrak{X}(G)$ be the left and right invariant vector field mappings, given by $L_X(g) = T_e(\mu_g) \cdot X$ and $R_X = T_e(\mu^g) \cdot X$, respectively. They are related by $L_X(g) = R_{\text{Ad}(g)X}(g)$. Their flows are given by

$$\text{Fl}_t^{L_X}(g) = g \cdot \exp(tX) = \mu^{\exp(tX)}(g), \qquad \text{Fl}_t^{R_X}(g) = \exp(tX) \cdot g = \mu_{\exp(tX)}(g).$$

Let $\kappa^l, \kappa^r :\in \Omega^1(G, \mathfrak{g})$ be the left and right Maurer–Cartan forms, given by $\kappa^l_g(\xi) = T_g(\mu_{g^{-1}}) \cdot \xi$ and $\kappa^r_g(\xi) = T_g(\mu^{g^{-1}}) \cdot \xi$, respectively. These are the inverses to L, R in the following sense: $L_g^{-1} = \kappa^l_g : T_gG \to \mathfrak{g}$ and $R_g^{-1} = \kappa^r_g : T_gG \to \mathfrak{g}$. They are related by $\kappa^r_g = \text{Ad}(g)\kappa^l_g : T_gG \to \mathfrak{g}$ and they satisfy the Maurer–Cartan equations $d\kappa^l + \frac{1}{2}[\kappa^l, \kappa^l]^\wedge = 0$ and $d\kappa^r - \frac{1}{2}[\kappa^r, \kappa^r]^\wedge = 0$.

The (exterior) derivative of the function $\text{Ad} : G \to GL(\mathfrak{g})$ can be expressed by

$$d\,\text{Ad} = \text{Ad} \cdot (\text{ad} \circ \kappa^l) = (\text{ad} \circ \kappa^r) \cdot \text{Ad},$$

which follows from $d\,\text{Ad}(T\mu_g.X) = \frac{d}{dt}|_0 \text{Ad}(g \cdot \exp(tX)) = \text{Ad}(g) \cdot \text{ad}(\kappa^l(T\mu_g \cdot X))$.

3.2. Analysis on Lie groups. Let V be a vector space. For $f \in C^\infty(G, V)$ we have $df \in \Omega^1(G; V)$, a 1-form on G with values in V. We define the *left derivative* $\delta f = \delta^l f : G \to L(\mathfrak{g}, V)$ of f by

$$\delta f(x) \cdot X := df \cdot T_e(\mu_x) \cdot X = (L_X f)(x) \quad \text{for } x \in G, \quad X \in \mathfrak{g}.$$

Result [27].

(1) *For $f \in C^\infty(G, \mathbb{R})$ and $g \in C^\infty(G, V)$ we have $\delta(f \cdot g) = f \cdot \delta g + \delta f \otimes g$, where we use $\mathfrak{g}^* \otimes V \to L(\mathfrak{g}, V)$.*

(2) *For $f \in C^\infty(G, V)$ we have $\delta\delta f(x)(X, Y) - \delta\delta f(x)(Y, X) = \delta f(x)([X, Y])$.*

(3) *Fundamental theorem of calculus: For $f \in C^\infty(G, V)$, $x \in G$, $X \in \mathfrak{g}$ we have*

$$f(x \cdot \exp(X)) - f(x) = \left(\int_0^1 \delta f(x \cdot \exp(tX))dt \right)(X).$$

(4) *Taylor expansion with remainder: For* $f \in C^\infty(G, V)$, $x \in G$, $X \in \mathfrak{g}$ *we have*

$$f(x \cdot \exp(X)) = \sum_{j=0}^{N} \frac{1}{j!}\delta^j f(x)(X^j) + \int_0^1 \frac{(1-t)^N}{N!}\delta^{N+1} f(x \cdot \exp(tX))dt(X^{N+1}).$$

(5) *For* $f \in C^\infty(G, V)$ *and* $x \in G$ *the formal Taylor series*

$$\mathrm{Tay}_x f = \sum_{j=0}^{\infty} \frac{1}{j!}\delta^j f(x) : \bigotimes \mathfrak{g} \to \mathbb{R}$$

factors to a linear functional on the universal enveloping algebra: $\mathcal{U}(\mathfrak{g}) \to \mathbb{R}$. *If for* $A \in \mathcal{U}(\mathfrak{g})$ *we denote by* L_A *the associated left invariant differential operator on* G, *we have* $\langle A, \mathrm{Tay}_x f\rangle = (L_A f)(x)$

3.3. Vector fields and differential forms. For $f \in C^\infty(G, \mathfrak{g})$ we get a smooth vector field $L_f \in \mathfrak{X}(G)$ by $L_f(x) := T_e(\mu_x) \cdot f(x)$. This describes an isomorphism $L : C^\infty(G, \mathfrak{g}) \to \mathfrak{X}(G)$. If $h \in C^\infty(G, V)$ then we have $L_f h(x) = dh(L_f(x)) = dh \cdot T_e(\mu_x) \cdot f(x) = \delta h(x) \cdot f(x)$, for which we write shortly $L_f h = \delta h \cdot f$.

For $g \in C^\infty(G, \wedge^k\mathfrak{g}^*)$ we get a k-form $L_g \in \Omega^k(G)$ by the prescription $(L_g)_x = g(x) \circ \wedge^k T_x(\mu_{x^{-1}})$. This gives an isomorphism $L : C^\infty(G, \wedge\mathfrak{g}) \to \Omega(G)$.

Result [27].

(1) *For* $f, g \in C^\infty(G, \mathfrak{g})$ *we have*

$$[L_f, L_g]_{\mathfrak{X}(G)} = L_{K(f,g)},$$

where $K(f, g)(x) := [f(x), g(x)]_\mathfrak{g} + \delta g(x) \cdot f(x) - \delta f(x) \cdot g(x)$, *or shorter* $K(f, g) = [f, g]_\mathfrak{g} + \delta g \cdot f - \delta f \cdot g$.

(2) *For* $g \in C^\infty(G, \wedge^k\mathfrak{g}^*)$ *and* $f_i \in C^\infty(G, \mathfrak{g})$ *we have* $L_g(L_{f_1}, \ldots, L_{f_k}) = g \cdot (f_1, \ldots, f_k)$.

(3) *For* $g \in C^\infty(G, \wedge^k\mathfrak{g}^*)$ *the exterior derivative is given by*

$$d(L_g) = L_{\delta^\wedge g + \partial^\mathfrak{g} \circ g},$$

where $\delta^\wedge g : G \to \wedge^{k+1}\mathfrak{g}^*$ *is given by*

$$\delta^\wedge g(x)(X_0, \ldots, X_k) = \sum_{i=0}^{k} (-1)^i \delta g(x)(X_i)(X_0, \ldots, \widehat{X_i}, \ldots, X_k),$$

and where $\partial^\mathfrak{g}$ *is the Chevalley differential on* $\wedge\mathfrak{g}^*$.

(4) *For* $g \in C^\infty(G, \wedge^k\mathfrak{g}^*)$ *and* $f \in C^\infty(G, \mathfrak{g})$ *the Lie derivative is given by*

$$\mathcal{L}_{L_f} L_g = L_{\mathcal{L}^\mathfrak{g}_f \circ g + \mathcal{L}^\delta_f g},$$

where

$$(\mathcal{L}^{\mathfrak{g}}_f g)(x)(X_1,\dots,X_k)=\sum_i(-1)^i g(x)([f(x),X_i],X_1,\dots,\widehat{X_i},\dots,X_k),$$
$$(\mathcal{L}^{\delta}_f g)(x)(X_1,\dots,X_k)=\delta g(x)(f(x))(X_1,\dots,X_k)+\sum_i(-1)^i g(x)(\delta f(x)(X_i),X_1,\dots,\widehat{X_i},\dots,X_k).$$

3.4. Multi vector fields and the Schouten–Nijenhuis bracket. Recall that on a manifold M the space of multi vector fields $\Gamma(\wedge TM)$ carries the Schouten–Nijenhuis bracket, given by

$$\text{(1)}\qquad [X_1\wedge\cdots\wedge X_p, Y_1\wedge\cdots\wedge Y_q] = \sum_{i,j}(-1)^{i+j}[X_i,Y_j]\wedge\cdots\widehat{X_i}\cdots\wedge X_p\wedge Y_1\wedge\cdots\widehat{Y_j}\cdots\wedge Y_q.$$

See [28] for a presentation along the lines used here. This bracket has the following properties: Let $U\in\Gamma(\wedge^u TM)$, $V\in\Gamma(\wedge^v TM)$, $W\in\Gamma(\wedge^w TM)$, and $f\in C^\infty(M,\mathbb{R})$. Then

$$[U,V]=-(-1)^{(u-1)(v-1)}[V,U],$$
$$[U,[V,W]]=[[U,V],W]+(-1)^{(u-1)(v-1)}[V,[U,W]],$$
$$[U,V\wedge W]=[U,V]\wedge W+(-1)^{(u-1)v}V\wedge[U,W],$$
$$[f,U]=-\bar{\imath}(df)U,$$

where $\bar{\imath}(df)$ is the insertion operator $\wedge^k TM\to\wedge^{k-1}TM$, the adjoint of $df\wedge(\ )$: $\wedge^l T^*M\to\wedge^{l+1}T^*M$.

For a Lie group G we have an isomorphism $L: C^\infty(G,\wedge\mathfrak{g})\to\Gamma(\wedge TG)$ which is given by $L(u)_x=\wedge T(\mu_x)\cdot u(x)$, via left trivialization. For $u\in C^\infty(G,\wedge^u\mathfrak{g})$ we have $\delta u: G\to L(\mathfrak{g},\wedge^u\mathfrak{g})=\mathfrak{g}^*\otimes\wedge^u\mathfrak{g}$, and with respect to the one component in $\mathfrak{g}^*$ we can consider the insertion operator $\bar{\imath}(\delta u(x)):\wedge^k\mathfrak{g}\to\wedge^{k+u}\mathfrak{g}$. In more detail: if $u=f\cdot U$ for $f\in C^\infty(G,\mathbb{R})$ and $U\in\wedge^u\mathfrak{g}$, then we put $\bar{\imath}(\delta f(x)\cdot U)V=U\wedge\bar{\imath}(\delta f(x))(V)$.

For the Lie algebra $\mathfrak{g}$ we also have the algebraic Schouten–Nijenhuis bracket $[\ ,\ ]^{\mathfrak{g}}$: $\wedge^p\mathfrak{g}\times\wedge^q\mathfrak{g}\to\wedge^{p+q-1}\mathfrak{g}$ which is given by formula (1), applied to this purely algebraic situation.

Proposition. *For $u\in C^\infty(G,\wedge^u\mathfrak{g})$ and $v\in C^\infty(G,\wedge^v\mathfrak{g})$ the Schouten–Nijenhuis bracket is given by*

$$\text{(2)}\qquad [L(u),L(v)]=L([u,v]^{\mathfrak{g}}-\bar{\imath}(\delta u)(v)+(-1)^{(u-1)(v-1)}\bar{\imath}(\delta v)(u)).$$

Proof. This follows from formula (1), applied to

$$[L(f\cdot X_1\wedge\cdots\wedge X_p), L(g\cdot Y_1\wedge\cdots\wedge Y_q)],$$

where $f,g\in C^\infty(G,\mathbb{R})$ and $X_i,Y_j\in\mathfrak{g}$, and then by applying 3.3(1). □

4. Lie–Poisson groups and double groups

4.1. Lie–Poisson groups. A *Poisson structure* on a Lie group is a tensor field $\Lambda \in \Gamma(\wedge^2 TG)$ such that $\{f, g\} := \langle df \wedge dg, \Lambda\rangle$ defines a Lie bracket on $C^\infty(G, \mathbb{R})$. If we let $\Lambda = L(\lambda)$ for $\lambda \in C^\infty(G, \wedge^2\mathfrak{g})$ in the notation of 3.4, then Λ is a Poisson structure if and only if for the Schouten bracket we have $[\Lambda, \Lambda] = 0$. By Proposition 3.4 this is equivalent to

(1) $\quad [\lambda(g), \lambda(g)]^{\mathfrak{g}} = 2\bar{\imath}(\delta\lambda(g))(\lambda(g)) \quad$ for all $g \in G$.

A *Lie–Poisson* group [11] is a Lie group G together with a Poisson structure $\Lambda \in \Gamma(\wedge^2 TG)$ such that the multiplication $\mu : G \times G \to G$ is a Poisson map, i.e. the pull back mapping $\mu^* : C^\infty(G, \mathbb{R}) \to C^\infty(G \times G, \mathbb{R})$ is a homomorphism for the Poisson brackets. This is equivalent to any of the following properties (2)–(6) for $p = 2$ (see [21]). Such a 2-vector field Λ is also called a *Lie–Poisson structure.*

Lemma. *For $\Lambda \in \Gamma(\wedge^p TG)$ the following assertions* (2)–(6) *are equivalent:*

(2) *Λ is multiplicative in the sense that*

$$\Lambda(gh) = \wedge^p T(\mu_g) \cdot \Lambda(h) + \wedge^p T(\mu^h) \cdot \Lambda(g) \quad \text{for all } g, h \in G.$$

(3) *(Assuming that G is connected) $\Lambda(e) = 0$ and the Schouten bracket $\mathcal{L}_{L_X}\Lambda = [L_X, \Lambda]$ is left invariant for each left invariant vector field L_X on G.*

(4) *(Assuming that G is connected) $\Lambda(e) = 0$ and the Schouten bracket $\mathcal{L}_{R_X}\Lambda = [R_X, \Lambda]$ is right invariant for each right invariant vector field R_X on G.*

(5) *If we let $\Lambda = L(\lambda)$ for $\lambda \in C^\infty(G, \wedge^p \gamma)$ in the notation of* 3.4, *then*

$$\lambda(gh) = \wedge^p \mathrm{Ad}(h^{-1}) \cdot \lambda(g) + \lambda(h) \quad \text{for all} \quad g, h \in G.$$

This has the following meaning: Consider the right semidirect product $G \ltimes \wedge^p\mathfrak{g}$ with multiplication $(x, U) \cdot (y, V) = (xy, \mathrm{Ad}(y^{-1})U + V)$. Then the above equation holds if and only if $x \mapsto (x, \lambda(x))$ is a homomorphism of Lie groups.

(6) *$\Lambda : G \to \wedge^p TG$ is a homomorphism of Lie groups, where $L : G \ltimes \wedge^p\mathfrak{g} \cong \wedge^p TG$.*

A Poisson structure Λ on G is a Lie–Poisson structure if and only if these conditions (2)–(6) *are satisfied for $p = 2$.*

Proof. For the proof of the equivalence of conditions (2)–(4) see [21], the equivalence to (5) and (6) is obvious.

We prove the last assertion. It follows from

$$\begin{aligned}
\{\mu^* f, \mu^* g\}_{G\times G}(x, y) &= \langle d(f \circ \mu) \wedge d(g \circ \mu), \Lambda(x) \otimes \Lambda(y)\rangle \\
&= (df(xy) \wedge dg(xy)) \cdot \wedge^2 T_{(x,y)}\mu \cdot (\Lambda(x), \Lambda(y)) \\
&= (df(xy) \wedge dg(xy)) \cdot \wedge^2(T_y(\mu_x) + T_x(\mu^y)) \cdot (\Lambda(x), \Lambda(y)) \\
&= (df(xy) \wedge dg(xy)) \cdot (\wedge^2 T_y(\mu_x)\Lambda(y) + \wedge^2 T_x(\mu^y)\Lambda(x))
\end{aligned}$$

compared with

$$(\mu^*\{f, g\}_G)(x, y) = (df \wedge dg) \cdot \Lambda(xy). \qquad \square$$

Note that if $\Lambda_1 : G \to \wedge^{p_1} TG$ and $\Lambda_2 : G \to \wedge^{p_2} TG$ are homomorphisms of groups with $\pi \circ \Lambda_i = \mathrm{Id}_G$, then their Schouten bracket $[\Lambda_1, \Lambda_2] : G \to \wedge^{p_1+p_2-1} TG$ has the same property. This follows from [21] and the equivalence to (6) from above.

4.2. Theorem [11]. *If (G, Λ) is a Lie–Poisson group with Lie algebra $\mathfrak{g}$ then by $b' : \mathfrak{g} \to \wedge^2\mathfrak{g}$ we get a Lie bialgebra structure on $\mathfrak{g}$, where $b'(X) = (\mathcal{L}_{L_X}\Lambda)(e) = \delta\lambda(e)X$, where $\mathcal{L}$ denotes the Lie derivative.*

If $(\mathfrak{g}, b, b')$ is a Lie bialgebra and G is a simply connected Lie group associated to $\mathfrak{g}$, then the cocycle b' integrates to a unique Lie–Poisson structure $\Lambda \in \Gamma(\wedge^2 TG)$ on G.

Proof. See [11,21] for other proofs. By conditions 4.1(5) and (6) any multiplicative 2-vector-field Λ is a homomorphism of Lie-groups

$$\begin{array}{ccc} G & \xrightarrow{\ \Lambda\ } & \wedge^2 TG \\ \Big\| & & \cong \uparrow \\ G & \xrightarrow{(\mathrm{Id}, \lambda)} & G \ltimes \wedge^2\mathfrak{g} \end{array}$$

and the induced Lie algebra homomorphism then is

$$\begin{aligned} T_e(\Lambda) \cdot X &:= (X, \mathcal{L}_{L_X}\Lambda(e)) \\ &= (X, \delta\lambda(e) \cdot X) \quad \text{(by Proposition 3.4(2))} \\ &= (\mathrm{Id}_{\mathfrak{g}}, b')(X), \end{aligned}$$

and conversely any 2-cocycle $b' : \mathfrak{g} \to \wedge^2\mathfrak{g}$ integrates to a Lie group homomorphism if G is supposed to be simply connected.

It remains to show that $b' : \wedge^2\mathfrak{g}^* \to \mathfrak{g}^*$ satisfies the Jacobi identity if and only if 4.1(1) holds. Let us take the left derivative δ at e of Eq. 4.1(1) and get

$$\begin{aligned} 0 &= 2[\delta\lambda(e), \lambda(e)]^{\mathfrak{g}} - 2\bar{\imath}(\delta^2\lambda(e))\lambda(e) - 2\bar{\imath}(\delta\lambda(e))\delta\lambda(e) \\ &= 0 - 0 - [\delta\lambda(e), \delta\lambda(e)]^{\mathrm{NR}}, \end{aligned}$$

so that the Nijenhuis–Richardson bracket of $b' = \delta\lambda(e) : \wedge^2\mathfrak{g}^* \to \mathfrak{g}^*$ with itself vanishes. This just means that b' is a Lie bracket on $\mathfrak{g}^*$, see [30].

For the converse note first that if $\Lambda : G \to \wedge^2 TG$ is a homomorphism of Lie groups then also the Schouten bracket $[\Lambda, \Lambda] : G \to \wedge^3 TG$ is a homomorphism of Lie groups. But if $b' = \delta\lambda(e)$ is a Lie bracket on $\mathfrak{g}^*$ then the computation above shows that $\delta([\lambda, \lambda]^{\mathfrak{g}} - 2\bar{\imath}(\delta\lambda)\lambda)(e) = 0$ so that the associated Lie algebra homomorphism is just $(\mathrm{Id}, 0) : \mathfrak{g} \to \mathfrak{g} \ltimes \wedge^3\mathfrak{g}$. But then $[\Lambda, \Lambda] = 0$. $\square$

4.3. Affine poisson structures. An affine Poisson structure on a Lie group G is a Poisson structure Λ such that Λ_l is a Lie–Poisson structure or equivalently Λ_r is a Lie–Poisson structure, where

(1) $\Lambda_l(g) = \Lambda(g) - T(\mu_g)\Lambda(e), \quad \Lambda_l = \Lambda - L_{\Lambda(e)},$

(2) $\Lambda_r(g) = \Lambda(g) - T(\mu^g)\Lambda(e), \quad \Lambda_r = \Lambda - R_{\Lambda(e)}.$

For a Poisson structure Λ we also have

(1a) $\Lambda_l = L(\lambda_l), \quad \lambda_l(g) = \lambda(g) - \lambda(e),$

(2a) $\Lambda_r = L(\lambda_r), \quad \lambda_r(g) = \lambda(g) - \mathrm{Ad}(g^{-1})\lambda(e),$

and Λ is an affine Poisson structure if and only if

(3) $\lambda(gh) = \mathrm{Ad}(h^{-1})\lambda(g) + \lambda(h) - \mathrm{Ad}(h^{-1})\lambda(e).$

4.4. Lie groups with exact Lie bialgebras. Let G be a Lie group with Lie algebra $\mathfrak{g}$. Suppose we have a solution $C \in \wedge^2\mathfrak{g}$ of the (mYBE), so that $b' = \partial C$ is a Lie bialgebra structure for $(\mathfrak{g}, \mathfrak{g}^*)$. Then we can write down explicitly the Lie–Poisson structure on any (even not connected) Lie group with Lie algebra $\mathfrak{g}$, as follows.

We consider $\Lambda_\pm : G \to \wedge^2 TG$ qiven by $\Lambda_\pm(g) := T(\mu_g)C \pm T(\mu^g)C$. Then obviously Λ_- is multiplicative and Λ_+ is affine with $(\Lambda_+)_l = \Lambda_-$ and $(\Lambda_+)_r = -\Lambda_-$. In the notation of (4.1) we have $\lambda_\pm(g) = C \pm \mathrm{Ad}(g^{-1})C$, and

$$b'_\pm(X) = \delta\lambda_\pm(e)X = \pm(\delta(\wedge^2(\mathrm{Ad} \circ \mathrm{Inv}))(e)X)C = \mp\mathrm{ad}(X)C = \mp(\partial_b C)(X),$$

and since C satisfies (mYBE), the tensor fields $\Lambda_\pm$ are Poisson structures.

4.5. Manin decompositions and Lie–Poisson structures. Let $\mathfrak{g} = \mathfrak{g}_+ \oplus \mathfrak{g}_-$ be a Manin decomposition of a metrical Lie algebra $\mathfrak{g}$, and let $\mathrm{pr}_\pm : \mathfrak{g} \to \mathfrak{g}_\pm$ be the corresponding projections. Then by 2.19 the operator $R = \mathrm{pr}_+ - \mathrm{pr}_-$ is a solution of (1-mYBE) $B_R + b = 0$.

So by (4.4) a Manin decomposition defines a canonically associated Lie–Poisson structure on each (even not connected) Lie group G with Lie algebra $\mathfrak{g}$, as follows: Let $C = R \circ g^{-1} \in \wedge\mathfrak{g}$ be the associated exact bialgebra structure, and consider $\Lambda_\pm : G \to \wedge^2 TG$ qiven by

(1) $\Lambda_\pm(g) := T(\mu_g)C \pm T(\mu^g)C.$

Then in the notation of (4.1) we have $\lambda_\pm(g) = C \pm \mathrm{Ad}(g^{-1})C$, and $b'_\pm(X) = \delta\lambda_\pm(e)X = \pm\wedge^2(\delta(\mathrm{Ad}\circ\mathrm{Inv})(e)X)C = \mp\mathrm{ad}(X)C = \mp(\partial_b C)(X)$. The tensor field Λ_- is a real analytic Lie–Poisson structure and Λ_+ is a real analytic affine Poisson structure with $(\Lambda_+)_l = \Lambda_-$ and $(\Lambda_+)_r = -\Lambda_-$. Since $\Lambda_+(e) = C$ is non-degenerate, the affine Poisson structure Λ_+ is non-degenerate on an open subset of G. If G is connected, this open subset is also dense since the real analytic Poisson structure cannot be degenerate on an open subset.

We shall investigate this kind of structure in much more details below.

4.6. Gauss decompositions and Lie–Poisson structures. Let G be a Lie group with a metrical Lie algebra $(\mathfrak{g}, g)$. From 2.20 we know that any solution R of the R-matrix equation can be described as follows. There is a Gauss decomposition $\mathfrak{g} = \mathfrak{g}_+ \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_-$ with $\mathfrak{g}_\pm$ isotropic and dual to each other, and with g non-degenerate on $\mathfrak{g}_0$. Let $\mathrm{pr}_{\pm,0} : \mathfrak{g} \to \mathfrak{g}_{\pm,0}$ be the orthogonal projections. Then R is of the following form:

(1) $$R = \mathrm{pr}_+ \oplus (R_0 \circ \mathrm{pr}_0) \oplus (-\mathrm{pr}_-),$$

where R_0 is a solution of (1-mYBE) on $\mathfrak{g}_0$ without eigenvalues 1 or -1 (without fixed points).

Let X_i be a basis of $\mathfrak{g}_+$ with Y_i the dual basis of $\mathfrak{g}_-$, and let Z_j be an orthonormal basis of $\mathfrak{g}_0$, all with respect to the inner product g on $\mathfrak{g} = \mathfrak{g}_+ \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_-$. Let $R_0(Z_j) = \sum_k R_j^k Z_k = \sum_k C^{kj} Z_k$ be the (skew symmetric) matrix representation of R_0 with respect to the basis Z_j. Then

$$\mathrm{pr}_+(U) = \sum_i X_i . g(U, Y_i),$$
$$\mathrm{pr}_0(U) = \sum_j Z_j . g(U, Z_j),$$
$$\mathrm{pr}_-(U) = \sum_i Y_i . g(U, X_i),$$

so that

(2) $$R = \mathrm{pr}_+ - \mathrm{pr}_- + (R_0 \circ \mathrm{pr}_0) = \left(\sum_i X_i \wedge Y_i + \sum_{j,k} R_j^k Z_k \otimes Z_j \right) \circ g$$
$$C := R \circ g^{-1} = \sum X_i \wedge Y_i + \sum_{j<k} C^{jk} Z_j \wedge Z_k.$$

Let us consider $\Lambda_\pm : G \to \wedge^2 TG$ qiven by

(3) $$\Lambda_\pm(g) := T(\mu_g)C \pm T(\mu^g)C.$$

Then in the notation of (4.1) we have $\lambda_\pm(g) = C \pm \mathrm{Ad}(g^{-1})C$, and

$$b'_\pm(X) = \delta\lambda_\pm(e)X = \pm(\delta(\mathrm{Ad} \circ \mathrm{Inv})(e)X)C = \mp\mathrm{ad}(X)C = \mp(\partial_b C)(X).$$

Since R was a solution of (1-YBE) the tensor field Λ_- is a real analytic Lie–Poisson structure and Λ_+ is a real analytic affine Poisson structure with $(\Lambda_+)_l = \Lambda_-$ and $(\Lambda_+)_r = -\Lambda_-$. Since $\Lambda_+(e) = C$ is non-degenerate, the affine Poisson structure Λ_+ is non-degenerate on an open subset of G. If G is connected this open subset is also dense since the real analytic Poisson structure cannot be degenerate on an open subset.

5. Explicit formulas for Poisson structures on double Lie groups

5.1. The setting. It turns out that in the situation of 4.5 one can get very useful explicit formulae. Let us explain this setting once more, which will be used for the rest of this paper.

Let G be any Lie group with a metrical Lie algebra $(\mathfrak{g}, \gamma)$ and suppose that it admits a Manin decomposition $(\mathfrak{g} = \mathfrak{g}_+ \oplus \mathfrak{g}_-, \gamma)$. Let $\mathrm{pr}_\pm : \mathfrak{g} \to \mathfrak{g}_\pm$ be the corresponding projections. By 2.20 the operator $R = \mathrm{pr}_+ - \mathrm{pr}_-$ is a solution of (1-mYBE) $B_R + b = 0$.

Simplified notation. In order to compactify the notation we will use the following shorthand, in the rest of this paper: For $U \in \bigotimes^p \mathfrak{g}$ etc. and for $g \in G$ we let

$$gU = g \cdot U = \bigotimes^p T(\mu_g)U, \qquad Ug = U \cdot g = \bigotimes^p T(\mu^g)U.$$

Let X_i be a basis of $\mathfrak{g}_+$ with Y_i the dual basis of $\mathfrak{g}_-$ with respect to the inner product γ on $\mathfrak{g} = \mathfrak{g}_+ \oplus \mathfrak{g}_-$. Then

$$\mathrm{pr}_+(Z) = \sum_i \gamma(Z, Y_i) \cdot X_i = \left(\sum_i Y_i \otimes X_i\right) \gamma(Z),$$

$$\mathrm{pr}_-(Z) = \sum_i \gamma(Z, X_i) Y_i = \left(\sum_i X_i \otimes Y_i\right) \gamma(Z),$$

so that

$$\mathrm{pr}_+ = \left(\sum_i Y_i \otimes X_i\right) \circ \gamma = C_+ \circ \gamma,$$

$$\mathrm{pr}_- = \left(\sum_i X_i \otimes Y_i\right) \circ \gamma = C_- \circ \gamma,$$

where

$$C_+ = \sum_i Y_i \otimes X_i, \quad C_- = \sum_i X_i \otimes Y_i.$$

Then we have

$$\text{(1)} \qquad R = \mathrm{pr}_+ - \mathrm{pr}_- = \left(\sum_i Y_i \wedge X_i\right) \circ \gamma,$$

$$C = R \circ \gamma^{-1} = C_+ - C_- = \sum Y_i \wedge X_i.$$

Then we consider $\Lambda_\pm : G \to \wedge^2 TG$ qiven by (note the factor $\frac{1}{2}$)

$$\text{(2)} \qquad \Lambda_\pm(g) := \tfrac{1}{2}(gC \pm Cg).$$

Then in the notation of (4.1) we have $\lambda_\pm(g) = \frac{1}{2}(C \pm \mathrm{Ad}(g^{-1})C)$, and $b'_\pm(X) = \delta\lambda_\pm(e)X = \pm\frac{1}{2} \wedge^2 (\delta(\mathrm{Ad} \circ \mathrm{Inv})(e)X)C = \mp\frac{1}{2}\mathrm{ad}(X)C = \mp\frac{1}{2}(\partial_b C)(X)$. The tensor field Λ_- is a real analytic Lie–Poisson structure and Λ_+ is a real analytic affine Poisson structure with $(\Lambda_+)_l = \Lambda_-$ and $(\Lambda_+)_r = -\Lambda_-$. Since $\Lambda_+(e) = C$ is non-degenerate, the affine Poisson structure Λ_+ is non-degenerate on an open subset of G. If G is connected, this open subset is also dense since the real analytic Poisson structure cannot vanish on an open subset.

5.2. Lemma. *In the setting of* 5.1 *we have:*

$$(1)\qquad \Lambda_+(a) = aC_+ - C_-a = C_+a - aC_- = \sum_i (aY_i \otimes aX_i - X_ia \otimes Y_ia) = \sum_i (Y_ia \otimes X_ia - aX_i \otimes aY_i)$$

$$(2)\qquad \Lambda_-(a) = aC_+ - C_+a = C_-a - aC_- = \sum_i (aY_i \otimes aX_i - Y_ia \otimes X_ia) = \sum_i (X_ia \otimes Y_ia - aX_i \otimes aY_i)$$

Proof. The tensor fields do not look skew symmetric but observe that

$$(3)\qquad aC_+ + aC_- = C_+a + C_-a.$$

This is equivalent to $C_+ + C_- = \bigotimes^2 \mathrm{Ad}(a^{-1})(C_+ + C_-)$ which, when composed with γ, in $L(\mathfrak{g}, \mathfrak{g})$ just says that $\mathrm{Id}_\mathfrak{g} = \mathrm{pr}_+ + \mathrm{pr}_- = \mathrm{Ad}(a^{-1})\mathrm{Id}_\mathfrak{g}\mathrm{Ad}(a)$. Using (5) we have

$$\begin{aligned}
\Lambda_+(a) &= \tfrac{1}{2}(aC + Ca) = \tfrac{1}{2}(aC_+ - aC_- + C_+a - C_-a)\\
&= C_+a - aC_- = aC_+ - C_-a,\\
\Lambda_-(a) &= \tfrac{1}{2}(aC - Ca) = \tfrac{1}{2}(aC_+ - aC_- - C_+a + C_-a)\\
&= aC_+ - C_+a = C_-a - aC_-. \qquad \square
\end{aligned}$$

5.3. The subgroups and the Poisson structures. In the setting of 5.1 we consider now the Lie subgroups $G_\pm \subset G$ corresponding to the isotropic Lie subalgebras $\mathfrak{g}_\pm$, and we consider the mappings

$$\begin{aligned}
\varphi &: G_+ \times G_- \to G, \qquad \varphi(g, u) := g \cdot u \in G,\\
\psi &: G_- \times G_+ \to G, \qquad \psi(v, h) := v \cdot h \in G.
\end{aligned}$$

Both are diffeomorphisms on open neighborhoods of (e, e). We will use g, u and v, h as local 'coordinates' near e. So, we have, at least locally in an open neighborhood U of e in G, well-defined projections $p_l^+, p_r^+ : G \supset U \to G_+$ and $p_l^-, p_r^- : G \supset U \to G_-$ which play the role of momentum mappings:

$$\begin{aligned}
p_l^+(g \cdot u) &:= g, \quad p_r^+(v \cdot h) := h \in G_+,\\
p_l^-(v \cdot h) &:= v, \quad p_r^-(g \cdot u) := u \in G_-.
\end{aligned}$$

When φ (or equivalently ψ) is a global diffeomorphism (this is consistent for simply connected G with completeness of the the dressing vector fields; in these cases we will call G a *complete double group*) then the mappings $p_{l,r}^\pm$ are globally defined.

Remark. If the subgroup G_+ is compact then the double group G is complete. Similarly for G_-.

Indeed, there exists a G-invariant Riemann metric on the homogeneous space G/G_+. Then G acts on G/G_+ by isometries locally transitively, hence transitively. This means that $G = G_+ \cdot G_-$ globally and that $G_+ \cap G_-$ is finite.

5.4. Theorem. *In the setting above, the following tensor fields are Lie–Poisson structures on the group G_+ and G_-, respectively, corresponding to the Lie bialgebra structures on $\mathfrak{g}_+$ and $\mathfrak{g}_-$ induced from the Manin decomposition:*

$$(1)\qquad \begin{aligned} \Lambda^{G_+}(g) &= g((\mathrm{Id}_{\mathfrak{g}} \otimes \mathrm{Ad}^G(g^{-1})\mathrm{pr}_+\mathrm{Ad}^G(g))C_-) \in \wedge^2 TG_+ \\ &= g(-(\mathrm{Ad}^G(g^{-1}) \otimes \mathrm{pr}_+\mathrm{Ad}^G(g^{-1}))C_-) \\ &= \sum_i gX_i \otimes \mathrm{pr}_+(\mathrm{Ad}^G(g)Y_i)g \\ &= -\sum_i X_i g \wedge g\,\mathrm{pr}_+(\mathrm{Ad}^G(g^{-1})Y_i), \end{aligned}$$

$$(2)\qquad \begin{aligned} \Lambda^{G_-}(u) &= u((\mathrm{Id}_{\mathfrak{g}} \wedge \mathrm{Ad}^G(u^{-1})\mathrm{pr}_-\mathrm{Ad}^G(u))C_+) \in \wedge^2 TG_- \\ &= u(-(\mathrm{Ad}^G(u^{-1}) \otimes \mathrm{pr}_-\mathrm{Ad}^G(u^{-1}))C_+) \\ &= \sum_i uY_i \otimes \mathrm{pr}_-(\mathrm{Ad}^G(u)X_i)u \\ &= -\sum_i Y_i u \otimes u\,\mathrm{pr}_-(\mathrm{Ad}^G(u^{-1})X_i). \end{aligned}$$

The following tensor fields are non-degenerate Poisson structures on the groups $G_+ \times G_-$ and $G_- \times G_+$, respectively.

$$(3)\qquad \Lambda_+^{\varphi}(g,u) = \Lambda^{G_+}(g) + \Lambda^{G_-}(u) + \sum_i Y_i u \wedge gX_i \in \wedge^2 T(G_+ \times G_-),$$

$$(4)\qquad \Lambda_+^{\psi}(v,h) = -\Lambda^{G_+}(h) - \Lambda^{G_-}(v) + \sum_i vY_i \wedge X_i h \in \wedge^2 T(G_- \times G^+).$$

Moreover they are related to the affine Poisson structures on G, i.e., we have

$$(5)\qquad \wedge^2 T\varphi \cdot \Lambda_+^{\varphi} = \Lambda_+ \circ \varphi, \qquad \wedge^2 T\psi \cdot \Lambda_+^{\psi} = \Lambda_+ \circ \psi.$$

The following tensor fields are Lie–Poisson structures on the groups $G_+ \times G_-$ and $G_- \times G_+$, respectively:

$$(6)\qquad \Lambda_-^{\varphi}(g,u) = -\Lambda^{G_+}(g) + \Lambda^{G_-}(u) \in \wedge^2 T(G_+ \times G_-),$$

$$(7)\qquad \Lambda_-^{\psi}(v,h) = -\Lambda^{G_+}(h) + \Lambda^{G_-}(v) \in \wedge^2 T(G_- \times G_+).$$

Moreover they are related to the Lie–Poisson structure on G which corresponds to C, i.e. we have

$$(8)\qquad T\varphi \cdot \Lambda_-^{\varphi} = \Lambda_- \circ \varphi, \qquad T\psi \cdot \Lambda_-^{\psi} = \Lambda_- \circ \psi.$$

Proof. Using 5.2(1) we have

$$\begin{aligned}\Lambda_+(gu) &= \sum_i (guY_i \otimes guX_i - X_i gu \otimes Y_i gu) \\ &= \sum_i g(\mathrm{Ad}(u)Y_i \otimes \mathrm{Ad}(u)X_i - \mathrm{Ad}(g^{-1})X_i \otimes \mathrm{Ad}(g^{-1})Y_i)u \\ &= \sum_i g(\mathrm{Ad}(u)Y_i \otimes \mathrm{pr}_-(\mathrm{Ad}(u)X_i) + \mathrm{Ad}(u)Y_i \otimes \mathrm{pr}_+(\mathrm{Ad}(u)X_i) \\ &\quad - \mathrm{Ad}(g^{-1})X_i \otimes \mathrm{pr}_-(\mathrm{Ad}(g^{-1})Y_i) - \mathrm{Ad}(g^{-1})X_i \otimes \mathrm{pr}_+(\mathrm{Ad}(g^{-1})Y_i))u.\end{aligned}$$

In $L(\mathfrak{g}, \mathfrak{g})$ we have (compare with 5.1(1))

$$\begin{aligned}&\left(\sum_i \mathrm{Ad}(u)Y_i \otimes \mathrm{pr}_+(\mathrm{Ad}(u)X_i)\right) \circ \gamma \\ &\quad = \mathrm{pr}_+ \circ \mathrm{Ad}(u) \circ \mathrm{pr}_+ \circ \mathrm{Ad}(u^{-1}) \\ &\quad = \mathrm{pr}_+ \circ \mathrm{Ad}(u) \circ (\mathrm{Id}_{\mathfrak{g}} - \mathrm{pr}_-) \circ \mathrm{Ad}(u^{-1}) \\ &\quad = \mathrm{pr}_+ - \mathrm{pr}_+ \circ \mathrm{Ad}(u) \circ \mathrm{pr}_- \circ \mathrm{Ad}(u^{-1}) = \mathrm{pr}_+ - 0,\end{aligned}$$

for $\mathrm{pr}_+ \circ \mathrm{Ad}(u) \circ \mathrm{pr}_- = 0$ since $u \in G_-$. Thus we get

$$\sum_i \mathrm{Ad}(u)Y_i \otimes \mathrm{pr}_+(\mathrm{Ad}(u)X_i) = \sum_i Y_i \otimes X_i$$

and similarly we obtain

$$\sum_i \mathrm{Ad}(g^{-1})X_i \otimes \mathrm{pr}_-(\mathrm{Ad}(g^{-1})Y_i) = \sum_i X_i \otimes Y_i,$$

so that

$$\begin{aligned}\Lambda_+(gu) &= g\left(\sum_i uY_i \otimes \mathrm{pr}_-(\mathrm{Ad}(u)X_i)u\right) + g\left(\sum_i Y_i \wedge X_i\right) \\ &\quad - \left(\sum_i X_i g \otimes g\,\mathrm{pr}_+(\mathrm{Ad}(g^{-1})Y_i)\right)u \\ &= T_{(g,u)}\varphi\left(\Lambda_{G_-}(u) + \Lambda_{G_+}(g) + \sum_i Y_i u \wedge gX_i\right),\end{aligned}$$

which proves (3) and part of (5) In a similar way one proves (4) and the other part of (5).

Next we check that the two expressions for Λ^{G_+} in (1) are the same. We have to show that the following expression vanishes:

$$\begin{aligned}&\sum_i gX_i \otimes \mathrm{pr}_+(\mathrm{Ad}(g)Y_i)g + \sum_i X_i g \otimes g\,\mathrm{pr}_+(\mathrm{Ad}(g^{-1})Y_i) \\ &\quad = g\left(\sum_i X_i \otimes \mathrm{Ad}(g^{-1})\mathrm{pr}_+(\mathrm{Ad}(g)Y_i) + \sum_i \mathrm{Ad}(g^{-1})X_i \otimes \mathrm{pr}_+(\mathrm{Ad}(g^{-1})Y_i)\right).\end{aligned}$$

The term in brackets, composed with γ from the right, is the following endomorphism of $\mathfrak{g}$:

$$\begin{aligned}
&\mathrm{Ad}(g^{-1})\mathrm{pr}_+\mathrm{Ad}(g)\mathrm{pr}_- + \mathrm{pr}_+\mathrm{Ad}(g^{-1})\mathrm{pr}_-\mathrm{Ad}(g)\\
&\quad = \mathrm{Ad}(g^{-1})\mathrm{pr}_+\mathrm{Ad}(g)(\mathrm{Id}-\mathrm{pr}_+) + \mathrm{pr}_+\mathrm{Ad}(g^{-1})(\mathrm{Id}-\mathrm{pr}_+)\mathrm{Ad}(g)\\
&\quad = \mathrm{Ad}(g^{-1})\mathrm{pr}_+\mathrm{Ad}(g) - \mathrm{Ad}(g^{-1})\mathrm{pr}_+\mathrm{Ad}(g)\mathrm{pr}_+ + \mathrm{pr}_+ - \mathrm{pr}_+\mathrm{Ad}(g^{-1})\mathrm{pr}_+\mathrm{Ad}(g)\\
&\quad = \mathrm{Ad}(g^{-1})\mathrm{pr}_+\mathrm{Ad}(g) - \mathrm{pr}_+ + \mathrm{pr}_+ - \mathrm{Ad}(g^{-1})\mathrm{pr}_+\mathrm{Ad}(g) = 0,
\end{aligned}$$

since $\mathrm{Ad}(g^{-1})\mathfrak{g}_+ \subset \mathfrak{g}_+$ and $\mathrm{pr}_+|\mathfrak{g}_+ = \mathrm{Id}$. In the same way one shows that the the two expressions for Λ^{G_-} in (2) coincide, and similar computations show that all expressions in (1) and (2) are indeed skew-symmetric (which is clear from the beginning).

Next we show that Λ^{G_+} is multiplicative. We have the following chain of equivalent assertions:

$$\begin{aligned}
&\Lambda^{G_+}(gh) = g\Lambda^{G_+}(h) + \Lambda^{G_+}(g)h,\\
&(gh)^{-1}\Lambda^{G_+}(gh) = h^{-1}\Lambda^{G_+}(h) + h^{-1}g^{-1}\Lambda^{G_+}(g)h,\\
&\sum_i X_i \otimes \mathrm{Ad}(gh)^{-1}\mathrm{pr}_+(\mathrm{Ad}(gh)Y_i)\\
&\quad = \sum_i X_i \otimes \mathrm{Ad}(h^{-1})\mathrm{pr}_+(\mathrm{Ad}(h)Y_i)\\
&\qquad + \sum_i \mathrm{Ad}(h^{-1})X_i \otimes \mathrm{Ad}(gh)^{-1}\mathrm{pr}_+(\mathrm{Ad}(g)Y_i),\\
&\mathrm{Ad}(gh)^{-1}\mathrm{pr}_+\mathrm{Ad}(gh)\mathrm{pr}_-\\
&\quad = \mathrm{Ad}(h^{-1})\mathrm{pr}_+\mathrm{Ad}(h)\mathrm{pr}_- + \mathrm{Ad}(gh)^{-1}\mathrm{pr}_+\mathrm{Ad}(g)\mathrm{pr}_-\mathrm{Ad}(h).
\end{aligned}$$

Both sides of the last equation vanish when applied to elements of $\mathfrak{g}_+$, and on elements of $\mathfrak{g}_-$ we may delete the rightmost pr_-, so this is equivalent to

$$\begin{aligned}
\mathrm{pr}_+\mathrm{Ad}(gh) &= \mathrm{Ad}(g)\mathrm{pr}_+\mathrm{Ad}(h) + \mathrm{pr}_+\mathrm{Ad}(g)\mathrm{pr}_-\mathrm{Ad}(h)\\
&= \mathrm{Ad}(g)\mathrm{pr}_+\mathrm{Ad}(h) + \mathrm{pr}_+\mathrm{Ad}(g)(\mathrm{Id}-\mathrm{pr}_+)\mathrm{Ad}(h)\\
&= \mathrm{Ad}(g)\mathrm{pr}_+\mathrm{Ad}(h) + \mathrm{pr}_+\mathrm{Ad}(gh) - \mathrm{pr}_+\mathrm{Ad}(g)\mathrm{pr}_+\mathrm{Ad}(h),
\end{aligned}$$

which is true since $\mathrm{Ad}(g)(\mathfrak{g}_+) \subset \mathfrak{g}_+$.

Finally we show that the group homomorphism $\Lambda^{G_+} : G_+ \to \wedge^2 TG_+$ is associated to the bialgebra structure given by the Lie bracket on $\mathfrak{g}_- \xrightarrow{g} (\mathfrak{g}_+)^*$. For that we consider, as explained in 4.1 and in the proof of 4.2:

$$(9)\qquad \lambda^{G_+}(g) = g^{-1}\Lambda^{G_+}(g) = \sum_i X_i \otimes \mathrm{Ad}(g^{-1})\mathrm{pr}_+(\mathrm{Ad}(g)Y_i),$$

$$\delta\lambda^{G_+}(e)X = 0 + \sum_i X_i \otimes \mathrm{pr}_+(\mathrm{ad}(X)Y_i),$$

$$\gamma(\delta\lambda^{G_+}(e)X, Y_k \otimes Y_l)$$

$$= \sum_i \gamma(X_i, Y_k)\gamma(\mathrm{pr}_+\mathrm{ad}(X)Y_i, Y_l)$$
$$= \gamma(\mathrm{pr}_+\mathrm{ad}(X)Y_k, Y_l) = \gamma(\mathrm{ad}(X)Y_k, \mathrm{pr}_+^* Y_l)$$
$$= \gamma([X, Y_k], \mathrm{pr}_- Y_l) = \gamma(X, [Y_k, Y_l]),$$

which we had to prove. Let us now investigate the Lie–Poisson structure on G. From 5.2(2) we have

$$\begin{aligned}\Lambda_-(gu) &= \sum_i (guY_i \otimes guX_i - Y_i gu \otimes X_i gu)\\ &= \sum_i g(\mathrm{Ad}(u)Y_i \otimes \mathrm{Ad}(u)X_i - \mathrm{Ad}(g^{-1})Y_i \otimes \mathrm{Ad}(g^{-1})X_i)u\\ &= \sum_i g(\mathrm{Ad}(u)Y_i \otimes \mathrm{pr}_-(\mathrm{Ad}(u)X_i) + \mathrm{Ad}(u)Y_i \otimes \mathrm{pr}_+\mathrm{Ad}(u)X_i\\ &\quad - \mathrm{pr}_-(\mathrm{Ad}(g^{-1})Y_i) \otimes \mathrm{Ad}(g^{-1})X_i - \mathrm{pr}_+(\mathrm{Ad}(g^{-1})Y_i) \otimes \mathrm{Ad}(g^{-1})X_i)u.\end{aligned}$$

In $L(\mathfrak{g}, \mathfrak{g})$ we again have

$$\begin{aligned}&\left(\sum_i \mathrm{Ad}(u)Y_i \otimes \mathrm{pr}_+\mathrm{Ad}(u)X_i\right) \circ \gamma\\ &\quad = \mathrm{pr}_+\mathrm{Ad}(u)\mathrm{pr}_+\mathrm{Ad}(u^{-1})\\ &\quad = \mathrm{pr}_+\mathrm{Ad}(u)(\mathrm{Id} - \mathrm{pr}_-)\mathrm{Ad}(u^{-1}) = \mathrm{pr}_+ - 0,\\ &-\left(\sum_i \mathrm{pr}_-(\mathrm{Ad}(g^{-1})Y_i) \otimes \mathrm{Ad}(g^{-1})X_i\right) \circ \gamma\\ &\quad = -\mathrm{Ad}(g^{-1})\mathrm{pr}_+\mathrm{Ad}(g)\mathrm{pr}_-^*\\ &\quad = -\mathrm{Ad}(g^{-1})\mathrm{pr}_+\mathrm{Ad}(g)\mathrm{pr}_+ = -\mathrm{pr}_+,\\ &-\left(\sum_i \mathrm{pr}_+(\mathrm{Ad}(g^{-1})Y_i) \otimes \mathrm{Ad}(g^{-1})X_i\right) \circ \gamma\\ &\quad = -\mathrm{Ad}(g^{-1})\mathrm{pr}_+\mathrm{Ad}(g)\mathrm{pr}_+^*\\ &\quad = -\left(\sum_i X_i \otimes \mathrm{Ad}(g^{-1})\mathrm{pr}_+(\mathrm{Ad}(g)Y_i)\right) \circ \gamma.\end{aligned}$$

Thus we get

$$\begin{aligned}\Lambda_-(gu) &= \sum_i g(\mathrm{Ad}(u)Y_i \otimes \mathrm{pr}_-(\mathrm{Ad}(u)X_i) - X_i \otimes \mathrm{Ad}(g^{-1})\mathrm{pr}_+(\mathrm{Ad}(g)Y_i))u\\ &= g\left(\sum_i uY_i \otimes \mathrm{pr}_-(\mathrm{Ad}(u)X_i)u\right) - \left(\sum_i gX_i \otimes \mathrm{pr}_+(\mathrm{Ad}(g)Y_i)g\right)u\\ &= g\Lambda^{G_-}(u) - \Lambda^{G_+}(g)u = T_{(g,u)}\varphi(\Lambda^{G_-}(u) - \Lambda^{G_+}(g)),\end{aligned}$$

which proves (6) and (8). All remaining statements can be proved analogously, or are obvious. □

5.5 Corollary. *In the situation of* 5.1 *we have:*

(1) *The Poisson structure Λ_+^{φ} on the direct product group $G_+^{\mathrm{op}} \times G_-$ is affine with*

$$(\Lambda_+^{\varphi})_r(g, u) = \Lambda^{G_+}(g) + \Lambda^{G_-}(u),$$
$$(\Lambda_+^{\varphi})_l(g, u) = \Lambda^{G_+}(g) + \Lambda^{G_-}(u) + \sum_i Y_i u \wedge X_i g - \sum_i u Y_i \wedge g X_i,$$

where the vector fields $g \mapsto gX_i, X_i g$ are left and right invariant with respect to the opposite group structure on G_+.

(2) *Moreover, the Lie–Poisson structure $(\Lambda_+^{\varphi})_r$ on $G_+^{\mathrm{op}} \times G_-$ is the dual Lie Poisson structure to Λ_- on G, i.e., it defines the Lie algebra structure on $\mathfrak{g}$.*

(3) *The Poisson structure Λ_+^{ψ} on the direct product group $G_-^{\mathrm{op}} \times G_+$ is affine with*

$$(\Lambda_+^{\psi})_r(v, h) = -\Lambda^{G_+}(h) - \Lambda^{G_-}(v),$$
$$(\Lambda_+^{\psi})_l(v, h) = -\Lambda^{G_+}(h) - \Lambda^{G_-}(v) + \sum_i Y_i v \wedge X_i h - \sum_i v Y_i \wedge h X_i,$$

where the vector fields $g \mapsto vY_i, Y_i v$ are left and right invariant with respect to the opposite group structure on G_-.

(4) *$(G_+, -\Lambda^{G_+})$ and (G_-, Λ^{G_-}) are Lie–Poisson subgroups of the Lie–Poisson group (G, Λ_-).*

(5) *The (local) projections from* 5.3

$$p_l^+, p_r^+ : (G, \Lambda_-) \to (G_+, -\Lambda^{G_+}), \qquad p_l^-, p_r^- : (G, \Lambda_-) \to (G_-, \Lambda^{G_-}),$$

are Poisson mappings.

(6) *The (local) projections from* 5.3

$$p_l^+ : (G, \Lambda_+) \to (G_+, \Lambda^{G_+}), \qquad p_r^+ : (G, \Lambda_+) \to (G_+, -\Lambda^{G_+}),$$
$$p_l^- : (G, \Lambda_+) \to (G_-, \Lambda^{G_-}), \qquad p_r^- : (G, \Lambda_+) \to (G_-, -\Lambda^{G_-})$$

are Poisson mappings.

(7) *The mapping $(G_+, \Lambda^{G_+}) \times (G, \Lambda_+) \to (G, \Lambda_+)$ given by $(g, a) \mapsto ga$ is a left Poisson action of a Lie–Poisson group.*

(8) *The mapping $(G, \Lambda_+) \times (G_-, \Lambda^{G_-}) \to (G, \Lambda_+)$ given by $(a, u) \mapsto ga$ is a right Poisson action of a Lie–Poisson group.*

(9) *The Lie–Poisson group dual to (G, Λ_-) is $G_+ \times G_-^{\mathrm{op}}$ with the Lie–Poisson structure $-(\Lambda_+^{\psi})_l$.*

Proof. On the direct product group $G_+^{\mathrm{op}} \times G_-$ the vector field $g \mapsto X_i g$ is *right* invariant, so expressions in (1) follows directly from 4.3 and the form 5.4(3) of Λ_+^{φ}. The Poisson structure $(\Lambda_+^{\varphi})_r$ is then visibly a Lie–Poisson structure on $G_+^{\mathrm{op}} \times G_-$, so $(\Lambda_+^{\varphi})_r$ is affine. The proof of (3) is similar.

For (2) we consider, as explained in 4.1 and in the proof of 4.2, see also the proof of 5.4(9):

$$\begin{aligned}\lambda^{\varphi}_{+,l}(g,u) &= \lambda^{G_+}(g) + \lambda^{G_-}(u)\\ &\quad + \sum_i \mathrm{Ad}(u^{-1})Y_i \wedge X_i - \sum_i Y_i \wedge \mathrm{Ad}(g^{-1})X_i,\\ \delta\lambda^{\varphi}_{+,l}(e,e)(X,Y) &= \delta\lambda^{G_+}(e)X + \delta\lambda^{G_-}(e)Y\\ &\quad - \sum_i [Y,Y_i]_{\mathfrak{g}_-} \wedge X_i + \sum_i Y_i \wedge [X,X_i]_{\mathfrak{g}_+}\\ &= b^{\mathfrak{g}_-} + b^{\mathfrak{g}_+} - \sum_i [Y,Y_i]_{\mathfrak{g}_-} \wedge X_i + \sum_i Y_i \wedge [X,X_i]_{\mathfrak{g}_+},\end{aligned}$$

where $X \in \mathfrak{g}_+$ and $Y \in \mathfrak{g}_-$. If we take this into the inner product with elements $Y_k \otimes Y_l$, $Y_k \otimes X_l$, etc., use 5.4(9) and proceed as there, the result follows.

Conditions (5)–(8) follow from the formulae for Λ_+ and Λ_- in the 'coordinates' (g,u) and (v,h), and from the fact that Λ^{G_+} and Λ^{G_-} are multiplicative.

Condition (9) is analogous to (2). □

5.6. Let us note finally that the decompositions 5.4(3) and (4) of the Poisson structure Λ_+ on $G \cong G_+ \times G_-$ are surprisingly rigid.

Theorem. *Suppose that a Poisson structure Λ on a manifold $H \times K$ which is a product of two Lie groups of equal dimension admits a decomposition*

$$\Lambda(h,k) = \Lambda^H(h) + \Lambda^K(k) + \sum_i Y_i^r(k) \wedge X_i^l(h) \in \wedge^2 T_{(h,k)}(H \times K),$$

where Λ^H and Λ^K are tensor fields on H and K, respectively, and where X_i^l are the left invariant vector fields and Y_i^r the right invariant vector fields on H and K, with respect to bases X_i of $\mathfrak{h}$ and Y_i of $\mathfrak{k}$.

Then Λ^H and Λ^K are affine Poisson structures on H and K, respectively, and (H, Λ^H), (K, Λ^K) is a dual pair of Lie–Poisson groups and Λ represents the 'symplectic' Poisson tensor on the corresponding group double.

Proof. The vanishing Schouten bracket $[\Lambda, \Lambda]$ yields

$$\begin{aligned} 0 = {} & [\Lambda^H, \Lambda^H] && \in \Gamma(\wedge^3 TH)\\ & + [\Lambda^K, \Lambda^K] && \in \Gamma(\wedge^3 TK)\\ & + 2\sum_i Y_i^r \wedge [X_i^l, \Lambda^H] - \sum_{ij} [Y_i, Y_j]^r \wedge X_i^l \wedge X_j^l && \in \mathfrak{X}(K) \otimes \Gamma(\wedge^2 TH)\\ & - 2\sum_i [Y_i^r, \Lambda^K] \wedge X_i^l + \sum_{ij} Y_i^r \wedge Y_j^r \wedge [X_i, X_j]^l && \in \Gamma(\wedge^2 TK) \otimes \mathfrak{X}(H). \end{aligned}$$

Each of the lines vanishes by itself: The first two lines then say that Λ^H and Λ^K are Poisson tensors on H and K, respectively. Using the structure constants c_m^{ij} of $\mathfrak{h}$ with respect to the basis X_i, and d_m^{ij} of $\mathfrak{k}$ with respect to Y_i, the last two lines can be rewritten as

$$\sum_m Y_m^r \wedge [X_m^l, \Lambda^H] = \frac{1}{2} \sum_{ijm} d_m^{ij} Y_m^r \wedge X_i^l \wedge X_j^l,$$

$$\sum_m [Y_m^r, \Lambda^K] \wedge X_m^l = \frac{1}{2} \sum_{ijm} Y_i^r \wedge Y_j^r \wedge c_m^{ij} X_m^l,$$

or by

$$[X_m^l, \Lambda^H] = \mathcal{L}_{X_m^l} \Lambda^H = \frac{1}{2} \sum_{ij} d_m^{ij} X_i^l \wedge X_j^l \in L(\wedge^2 \mathfrak{h}),$$

$$[Y_m^r, \Lambda^K] = \mathcal{L}_{Y_m^r} \Lambda^K = \frac{1}{2} \sum_{ij} c_m^{ij} Y_i^r \wedge Y_j^r \in R(\wedge^2 \mathfrak{k}).$$

These are just conditions (3) and (4) of 4.1 without the further assumption that $\Lambda^H(e) = 0$ or $\Lambda^K(e) = 0$, so we can conclude from there that Λ^H and Λ^K are affine Poisson structures, respectively. For their associated Lie–Poisson structures

$$(\Lambda^H)_r(h) = \Lambda^H(h) - \Lambda^H(e)h, \qquad (\Lambda^K)_l(k) = \Lambda^K(h) - k\Lambda^K(e)$$

we get

$$\mathcal{L}_{X_m^l}(\Lambda^H)_r = \mathcal{L}_{X_m^l} \Lambda^H = \frac{1}{2} \sum_{ij} d_m^{ij} X_i^l \wedge X_j^l,$$

$$\mathcal{L}_{Y_m^r}(\Lambda^K)_l = \mathcal{L}_{Y_m^r} \Lambda^K = \frac{1}{2} \sum_{ij} c_m^{ij} Y_i^r \wedge Y_j^r,$$

so that the Lie–Poisson structure $(\Lambda^H)_r$ corresponds to the cobracket

$$b'_{\mathfrak{h}} : \mathfrak{h} \to \wedge^2 \mathfrak{h}, \qquad b'_{\mathfrak{h}}(X_m) = \frac{1}{2} \sum_{ij} d_m^{ij} X_i \wedge X_j,$$

and the Lie–Poisson bracket $(\Lambda^K)_l$ corresponds to the cobracket

$$b'_{\mathfrak{k}} : \mathfrak{k} \to \wedge^2 \mathfrak{k}, \qquad b'_{\mathfrak{k}}(Y_m) = \frac{1}{2} \sum_{ij} c_m^{ij} Y_i \wedge Y_j.$$

Hence $b'_{\mathfrak{h}}$ is dual to the Lie bracket on $\mathfrak{k}$, and $b'_{\mathfrak{k}}$ is dual to the Lie bracket on $\mathfrak{h}$, with respect to the pairing $\gamma(X_i, Y_j) = \delta_{ij}$. □

6. Dressing actions and symplectic leaves

6.1. Lie algebroids. On every Poisson manifold (M, Λ) the Poisson tensor defines the mapping $T^*M \ni \alpha \mapsto \alpha^\sharp := \bar{\iota}_\alpha \Lambda \in TM$, and a Lie bracket on the space of 1-forms defined by

(1) $$\{\alpha, \beta\} := i_{\alpha^\sharp} d\beta - i_{\beta^\sharp} d\alpha + d i_\Lambda (\alpha \wedge \beta).$$

The mapping $(\ )^\sharp : \Omega^1(M) \to \mathfrak{X}(M)$ is then a homomorphism of Lie algebras,

(2) $$\{\alpha, \beta\}^\sharp = [\alpha^\sharp, \beta^\sharp];$$

this is also expressed by saying that Λ turns T^*M into a Lie algebroid with anchor mapping $(\ )^\sharp$.

6.2. The dressing action. Affine Poisson structures on a Lie group G may be characterized by the property that the left invariant 1-forms (or equivalently the right invariant ones) are closed with respect to the bracket 6.1(1).

Consequently, for an affine Poisson structure Λ on G the mappings

$$\lambda : \mathfrak{g}^* \to \mathfrak{X}(G), \qquad \lambda(X)(a) := -(aX)^\sharp,$$
$$\rho : \mathfrak{g}^* \to \mathfrak{X}(G), \qquad \rho(X)(a) := (Xa)^\sharp$$

are an anti-homomorphism and homomorphism of the Lie algebras $\mathfrak{g}_r^*$ and $\mathfrak{g}_l^*$, respectively, where $\mathfrak{g}_r^*$ is the dual space $\mathfrak{g}^*$ with the Lie bracket corresponding to Λ_r, and where $\mathfrak{g}_l^*$ corresponds to Λ_l. The fields $\lambda(X)$ are called *left dressing vector fields* on G, and the $\rho(X)$ are called *right dressing vector fields*. They may be considered as infinitesimal actions of the corresponding dual groups. We have seen such actions already in 5.5(4) and (5). If we can integrate this infinitesimal action to a global one, called the dressing action (if the dressing fields are complete), the affine Poisson group (G, Λ) will be called *complete*.

In any case, the left (or right) dressing vector fields generate the characteristic distribution of Λ, whose leaves are precisely the symplectic leaves of the Poisson structure Λ.

One believes that dressing actions describe 'hidden symmetries' of physical systems.

6.3. Theorem. *Let G be a Lie group with a metrical Lie algebra $(\mathfrak{g}, \gamma)$ which admits a Manin decomposition $\mathfrak{g} = \mathfrak{g}_+ \oplus \mathfrak{g}_-$. In the setting of* 5.1, *the dressing vector fields for the affine Poisson structures Λ_+ and Λ_- on G are the following:*

(1) $$\lambda_+(X_i)(a) = -\mathrm{pr}_+(\mathrm{Ad}(a)X_i)a, \qquad \rho_+(X_i)(a) = a\,\mathrm{pr}_+(\mathrm{Ad}(a)^{-1}X_i),$$
$$\lambda_+(Y_i)(a) = \mathrm{pr}_-(\mathrm{Ad}(a)Y_i)a, \qquad \rho_+(Y_i)(a) = -a\,\mathrm{pr}_-(\mathrm{Ad}(a)^{-1}Y_i).$$

(2) $$\lambda_-(X_i)(a) = \mathrm{pr}_-(\mathrm{Ad}(a)X_i)a, \qquad \rho_-(X_i)(a) = -a\,\mathrm{pr}_-(\mathrm{Ad}(a)^{-1}X_i),$$
$$\lambda_-(Y_i)(a) = -\mathrm{pr}_+(\mathrm{Ad}(a)Y_i)a, \qquad \rho_-(Y_i)(a) = a\,\mathrm{pr}_+(\mathrm{Ad}(a)^{-1}Y_i).$$

Proof. For instance, by 5.2,

$$\begin{aligned}\rho_+(X_i)(a) &= \bar{\iota}(\gamma(X_i a))\Lambda_+(a) = \bar{\iota}(\gamma(X_i a))\sum_j (Y_j a \otimes X_j a - aX_j \otimes aY_j)\\ &= \sum_j (\gamma(X_i a, Y_j a)X_j a - \gamma(X_i a, aX_j)aY_j)\\ &= X_i a - a\,\mathrm{pr}_-(\mathrm{Ad}(a^{-1})X_i = a(\mathrm{Ad}(a^{-1})X_i - \mathrm{pr}_-(\mathrm{Ad}(a^{-1})X_i))\\ &= a(\mathrm{pr}_+(\mathrm{Ad}(a^{-1})X_i)). \qquad \square\end{aligned}$$

6.4. Corollary. *The Poisson tensors $\Lambda_\pm$ may be written in the following alternative form:*

(1) $$\begin{aligned}\Lambda_+(a) &= \sum_i (aY_i \oplus \mathrm{pr}_+(\mathrm{Ad}(a)X_i)a - aX_i \otimes \mathrm{pr}_-(\mathrm{Ad}(a)Y_i)a)\\ &= \sum_i (Y_i a \oplus a\,\mathrm{pr}_+(\mathrm{Ad}(a^{-1})X_i) - X_i a \otimes a\,\mathrm{pr}_-(\mathrm{Ad}(a^{-1})Y_i)).\end{aligned}$$

(2) $$\begin{aligned}\Lambda_-(a) &= -\sum_i (aX_i \oplus \mathrm{pr}_+(\mathrm{Ad}(a)Y_i)a - aY_i \otimes \mathrm{pr}_-(\mathrm{Ad}(a)X_i)a)\\ &= -\sum_i (X_i a \otimes a\,\mathrm{pr}_-(\mathrm{Ad}(a^{-1})Y_i) - Y_i a \oplus a\,\mathrm{pr}_+(\mathrm{Ad}(a^{-1})X_i)).\end{aligned}$$

Proof. From the definition of $(\ )^{\sharp} : T^*G \to TG$ we have

$$\Lambda_+ = \sum_i (-aY_i \otimes \lambda_+(X_i) - aX_i \otimes \lambda_+(Y_i)), \text{ etc.} \qquad \square$$

6.5. Corollary [1].

(1) *The characteristic distributions $S_\pm$ of the Poisson structures $\Lambda_\pm$ may be described as follows:*

$$\begin{aligned}S_+(a) &= a(\mathrm{pr}_+(\mathrm{Ad}(a^{-1})\mathfrak{g}_+) + \mathrm{pr}_-\mathrm{Ad}(a^{-1})\mathfrak{g}_-)\\ &= (\mathrm{pr}_+(\mathrm{Ad}(a)\mathfrak{g}_+) + \mathrm{pr}_-\mathrm{Ad}(a)\mathfrak{g}_-)a,\\ S_-(a) &= a(\mathrm{pr}_-(\mathrm{Ad}(a^{-1})\mathfrak{g}_+) + \mathrm{pr}_+\mathrm{Ad}(a^{-1})\mathfrak{g}_-)\\ &= (\mathrm{pr}_-(\mathrm{Ad}(a)\mathfrak{g}_+) + \mathrm{pr}_+\mathrm{Ad}(a)\mathfrak{g}_-)a.\end{aligned}$$

In particular, $S_+(a) + S_-(a) = T_aG$.

(2) *The symplectic leaves of S_+ are the connected components of the intersections of orbits $G_+\,a\,G_- \cap G_-\,a\,G_+$, and the symplectic leaves of S_- are the connected components of the intersections of orbits $G_-\,a\,G_- \cap G_+\,a\,G_+$, for $a \in G$.*

(3) *The Poisson structure Λ_+ is non-degenerate precisely on the set $G_+G_- \cap G_-G_+$; so it is globally non-degenerate if and only if $G_+G_- = G$. In particular, if (G, Λ_+) is complete* 6.2 *then Λ_+ is non-degenerate.*

Proof. (1) follows directly from Theorem (6.3) since the dressing vector fields generate the characteristic distribution. To prove (2) observe that the tangent space to the intersection of orbits $G_+ a G_- \cap G_- a G_+$ at $a \in G$ is

$$\begin{aligned}
&(\mathfrak{g}_+ a + a\mathfrak{g}_-) \cap (\mathfrak{g}_- a + a\mathfrak{g}_+) \\
&\quad = a((\mathrm{Ad}(a^{-1})\mathfrak{g}_+ + \mathfrak{g}_-) \cap (\mathrm{Ad}(a^{-1})\mathfrak{g}_- + \mathfrak{g}_+)) \\
&\quad = a(\mathrm{pr}_+(\mathrm{Ad}(a^{-1})\mathfrak{g}_+) + \mathrm{pr}_-(\mathrm{Ad}(a^{-1})\mathfrak{g}_-)) = S_+(a),
\end{aligned}$$

so that the connected components of $G_+ a G_- \cap G_- a G_+$ are integral submanifolds of S_+. For S_- the proof is similar.

The intersection $G_+G_- \cap G_-G_+$ is an open and dense subset of G consisting, by (2), of points where Λ_+ is non-degenerate. If the orbit G_+aG_- meets $G_+G_- \cap G_-G_+$ then it is contained in G_+G_-, so $G_+G_- \cap G_-G_+$ consists of all points where Λ_+ is non-degenerate. □

6.6. On $M := G_+G_- \cap G_-G_+$ the Poisson structure Λ_+ is symplectic, so let us describe the associated symplectic form $\omega = (\Lambda_+)^{-1}$ in terms of the coordinates (g, u) and (v, h) introduced in (5.3). We will start by describing the dressing vector fields on the groups $(G_+ \times G_-, \Lambda_+^{\varphi})$ and $(G_- \times G_+, \Lambda_+^{\psi})$. In order to avoid problems of always having to tell which multiplication is opposite, and to use a notation which differs from that used in Theorem 6.3 we will write $(uX)^{\sharp}$ for the dressing vector field corresponding to the left invariant 1-form on $G_+ \times G^*$ represented by $\eta(g, u) = uX$ in the obvious way:

$$\gamma(uX, gX_i + uY_j) = \gamma(uX, uY_j) = \gamma(X, Y_j), \text{ etc.}$$

After easy calculations we get from 5.4(3) and (4):

Theorem. *In the situations above, the dressing vector fields are given by:*

(1) *On* $(G_+ \times G_-, \Lambda_+^{\varphi}(g, u))$:

$$\begin{aligned}
(X_i u)^{\sharp} &= gX_i - u\,\mathrm{pr}_-(\mathrm{Ad}(u^{-1})X_i), \\
(Y_i g)^{\sharp} &= -g\,\mathrm{pr}_+(\mathrm{Ad}(g^{-1})Y_i) - \mathrm{pr}_-(\mathrm{Ad}(g^{-1})Y_i)u, \\
(uX_i)^{\sharp} &= \mathrm{pr}_-(\mathrm{Ad}(u)X_i)u + g\,\mathrm{pr}_+(\mathrm{Ad}(u)X_i)u, \\
(gY_i)^{\sharp} &= \mathrm{pr}_+(\mathrm{Ad}(g)Y_i)g - Y_i u.
\end{aligned}$$

(2) *On* $(G_- \times G_+, \Lambda_+^{\psi}(v, h))$:

$$\begin{aligned}
(vX_i)^{\sharp} &= X_i h - \mathrm{pr}_-(\mathrm{Ad}(v)X_i)v, \\
(hY_i)^{\sharp} &= -\mathrm{pr}_+(\mathrm{Ad}(h)Y_i)h - v\,\mathrm{pr}_-(\mathrm{Ad}(h)Y_i), \\
(X_i v)^{\sharp} &= v\,\mathrm{pr}_-(\mathrm{Ad}(v^{-1})X_i) + \mathrm{pr}_+(\mathrm{Ad}(v^{-1})X_i)h, \\
(Y_i h)^{\sharp} &= h\,\mathrm{pr}_+(\mathrm{Ad}(h^{-1})Y_i) - vY_i.
\end{aligned}$$

Denote now $(X_i u)^{\wedge} = \varphi_*(X_u)^{\sharp} \in \mathfrak{X}(G_+G_-)$, *etc., and* $(vX_i)^{\wedge} = \psi_*(X_i u)^{\sharp} \in \mathfrak{X}(G_-G_+)$, *etc., and call them the* undressing vector fields. *They are given at the point* $a = gu = vh \in M = G_+G_- \cap G_-G_+ \subset G$ *by*

$$
\begin{aligned}
(3)\qquad & (X_i u)^\wedge = a\,\mathrm{pr}_+(\mathrm{Ad}(u^{-1})X_i), && (Y_i g)^\wedge = -Y_i a, \\
& (u X_i)^\wedge = a X_i, && (g Y_i)^\wedge = -\mathrm{pr}_-(\mathrm{Ad}(g)Y_i)a, \\
& (v X_i)^\wedge = \mathrm{pr}_+(\mathrm{Ad}(v)X_i)a && (h Y_i)^\wedge = -a Y_i, \\
& (X_i v)^\wedge = X_i a, && (Y_i h)^\wedge = -a\,\mathrm{pr}_-(\mathrm{Ad}(h^{-1})Y_i).
\end{aligned}
$$

Proof. We only prove (3), and only one example:

$$
\begin{aligned}
(X_i u)^\wedge &= \varphi_*(X_i u)^\sharp = \varphi_*(g X_i - u\,\mathrm{pr}_-(\mathrm{Ad}(u^{-1})X_i)) \\
&= g X_i u - g u\,\mathrm{pr}_-(\mathrm{Ad}(u^{-1})X_i) \\
&= g u(\mathrm{Ad}(u^{-1})X_i - \mathrm{pr}_-(\mathrm{Ad}(u^{-1})X_i)) \\
&= a\,\mathrm{pr}_+(\mathrm{Ad}(u^{-1})X_i). \qquad \square
\end{aligned}
$$

6.7. Corollary. *At points $a = gu = vh \in M = G_+G_- \cap G_-G_+ \subset G$ the affine Poisson structure is given by*

$$
\begin{aligned}
(1)\qquad \Lambda_+(a) &= \sum_i ((u X_i)^\wedge \otimes (h Y_i)^\wedge + (X_i v)^\wedge \otimes (Y_i g)^\wedge) \\
&= \sum_i ((X_i u)^\wedge \otimes (g Y_i)^\wedge - (Y_i h)^\wedge \otimes (v X_i)^\wedge).
\end{aligned}
$$

The associated symplectic structure ω may be written as

$$
\begin{aligned}
(2)\qquad \omega_a &= \sum_i ((u X_i) \otimes (h Y_i) + (X_i v) \otimes (Y_i g)) \\
&= \sum_i ((X_i u) \otimes (g Y_i) - (Y_i h) \otimes (v X_i)^\wedge),
\end{aligned}
$$

where we identify the 1-forms uX_i, etc., on $G_+ \times G_-$ and the 1-forms hY_i, etc., on $G_- \times G_+$ with 1-forms on M via the diffeomorphisms φ and ψ. Formally correct we should write $(\varphi^{-1})^*(uX_i)$, etc.

Proof. The form (1) of $\Lambda_+(a)$ can be checked by easy calculations. But (1) shows that we can construct $\Lambda_+(a)$ from $(uX_i)^\wedge = \bar{\imath}(uX_i)\Lambda_+$, etc., thus we can construct $\omega_a = \Lambda_+(a)^{-1}$ in the same way from the corresponding 1-forms uX_i. $\square$

6.8. Remark. We can write 6.7(2) in a more ‘coordinate free’ form:

$$
\begin{aligned}
(1)\qquad \omega &= \gamma(\mu^\varphi_{G_-} \overset{\otimes}{,} \mu^\psi_{G_+}) + \gamma(\theta^\psi_{G_-} \overset{\otimes}{,} \mu^\varphi_{G_+}) \\
&= \tfrac{1}{2}(\gamma(\theta^\varphi_{G_-} \overset{\wedge}{,} \mu^\varphi_{G_+}) + \gamma(\mu^\psi_{G_-} \overset{\wedge}{,} \theta^\psi_{G_+})),
\end{aligned}
$$

where $\mu^\varphi_{G_-} = (uX_i) \otimes Y_i$ is the left Maurer–Cartan form on G_- pushed via φ to $M = G_+G_- \cap G_-G_+ \subset G$, and where $\theta^\psi_{G_-} = (X_i v) \otimes Y_i$ is the right Maurer–Cartan form on G_- pushed via ψ to M, etc. This expression (1) should be compared with the corresponding formula in [1], or with formula 2.3(3) in [2] for the case of a cotangent bundle T^*G_+. So 6.7 is a generalization of these results in [2] to the case of a double group.

6.9. Recall now from (5.3) the projections $p_l^+, p_r^+ : G \supset U \to G_+$ and $p_l^-, p_r^- : G \supset U \to G_-$ which we get from inverting φ and ψ, respectively. For $a \in G$ and for b near e in G we then define

$$(1) \qquad \begin{aligned} \lambda_b^+(a) &:= p_r^+(ab^{-1}a^{-1})a, & \lambda_b^-(a) &:= p_r^-(ab^{-1}a^{-1})a, \\ \rho_b^+(a) &:= ap_l^+(ab^{-1}a^{-1}), & \rho_b^-(a) &:= ap_l^-(ab^{-1}a^{-1}). \end{aligned}$$

Theorem. *The mappings λ^+ and λ^- define left (local) actions of G on G, and ρ^+ and ρ^- define right (local actions), i.e.,*

$$(2) \qquad \begin{aligned} \lambda_b^+(\lambda_{b'}^+(a)) &= \lambda_{bb'}^+(a), & \lambda_b^-(\lambda_{b'}^-(a)) &= \lambda_{bb'}^-(a), \\ \rho_b^+(\rho_{b'}^+(a)) &= \rho_{b'b}^+(a), & \rho_b^-(\rho_{b'}^-(a)) &= \rho_{b'b}^-(a). \end{aligned}$$

The subgroup G_+ is invariant under λ^+ and ρ^+ while G_- is invariant under λ^- and ρ^-.

Moreover, the pairs λ^+, λ^-, and ρ^+, ρ^- commute:

$$(3) \qquad \begin{aligned} \lambda_b^+(\lambda_{b'}^-(a)) &= \lambda_{b'}^-(\lambda_b^+(a)) = p_r^+(a(b')^{-1}ba^{-1})ab^{-1}, \\ \rho_b^+(\rho_{b'}^-(a)) &= \rho_{b'}^-(\rho_b^+(a)) = (b')^{-1}ap_r^+(a(b')^{-1}ba^{-1}). \end{aligned}$$

Proof. (2) Assume $a(b')^{-1}a^{-1} = vh$ for $v \in G_-$ and $h \in G_+$ as usual, so that we have $\lambda_{b'}^+(a) = ha = v^{-1}a(b')^{-1}$. Then

$$\begin{aligned} \lambda_b^+(\lambda_{b'}^+(a)) &= p_r^+(hab^{-1}a^{-1}h^{-1})ha = p_r^+(hab^{-1}a^{-1})a \\ &= p_r^+(v^{-1}a(b')^{-1}b^{-1}a^{-1})a \\ &= p_r^+(a(bb')^{-1}a^{-1})a = \lambda_{bb'}^+(a). \end{aligned}$$

For the other actions, the proofs are similar.

(3) We shall prove only the first part. Put $a(b')^{-1}a^{-1} = gu$ for $g \in G_+$ and $u \in G_-$, so that $\lambda_{b'}^-(a) = ua = g^{-1}a(b')^{-1}$. Then

$$\begin{aligned} \lambda_b^+(\lambda_{b'}^-(a)) &= \lambda_b^+(ua) = p_r^+(uab^{-1}(ua)^{-1})ua \\ &= p_r^+(ab^{-1}b'a^{-1}g)g^{-1}a(b')^{-1} \\ &= p_r^+(ab^{-1}b'a^{-1})a(b')^{-1} = p_r^-(ab^{-1}b'a^{-1})^{-1}(ab^{-1}b'a^{-1})a(b')^{-1} \\ &= p_r^-(a(b')^{-1}ba^{-1})ab^{-1}. \end{aligned}$$

On the other hand, put $ab^{-1}a^{-1} = vh$, so that $\lambda_b^+(a) = ha = v^{-1}ab^{-1}$. Then

$$\begin{aligned} \lambda_{b'}^-(\lambda_b^+(a)) &= p_r^-(ha(b')^{-1}(ha)^{-1})ha = p_r^-(a(b')^{-1}ba^{-1}v)v^{-1}ab^{-1} \\ &= p_r^-(a(b')^{-1}ba^{-1})ab^{-1}. \qquad \square \end{aligned}$$

6.10. Theorem. *The infinitesimal actions for λ^+, λ^-, ρ^+, and ρ^- are the following, where $A, B \in \mathfrak{g} = \mathfrak{g}_+ \oplus \mathfrak{g}_-$ and $a \in G$:*

$$(1) \qquad \begin{aligned} \lambda_B^+(a) &:= -\mathrm{pr}^+(\mathrm{Ad}(a)B)a, & \lambda_B^-(a) &:= -\mathrm{pr}^-(\mathrm{Ad}(a)B)a, \\ \rho_B^+(a) &:= -a\,\mathrm{pr}^+(\mathrm{Ad}(a^{-1})B), & \rho_B^-(a) &:= -a\,\mathrm{pr}^-(\mathrm{Ad}(a^{-1})B). \end{aligned}$$

Furthermore, as usual for left and right actions, for $B, B' \in \mathfrak{g}$ we have

$$
\text{(2)} \quad \begin{array}{ll}
[\lambda^+_B, \lambda^+_{B'}] = -\lambda^+_{[B,B']}, & [\lambda^-_B, \lambda^-_{B'}] = -\lambda^-_{[B,B']}, \\
[\rho^+_B, \rho^+_{B'}] = \rho^+_{[B,B']}, & [\rho^-_B, \rho^-_{B'}] = \rho^-_{[B,B']}, \\
[\lambda^+_B, \lambda^-_{B'}] = 0, & [\rho^+_B, \rho^-_{B'}] = 0.
\end{array}
$$

Moreover,

$$
\text{(3)} \quad \begin{array}{ll}
\lambda_+(X_i) = \lambda^+_{X_i}, & \lambda_+(Y_i) = -\lambda^-_{Y_i}, \\
\rho_+(X_i) = -\rho^+_{X_i}, & \rho_+(Y_i) = -\rho^-_{Y_i}, \\
\lambda_-(X_i) = -\lambda^-_{X_i}, & \lambda_-(Y_i) = \lambda^+_{Y_i}, \\
\rho_-(X_i) = -\rho^-_{X_i}, & \rho_-(Y_i) = \rho^+_{Y_i},
\end{array}
$$

so that we can reconstruct the dressing actions from λ^+, λ^-, ρ^+, and ρ^-. For example, the (local) left dressing action for Λ_+ is given by

$$
\begin{aligned}
&G_+ \times (G_-)^{\mathrm{op}} \times G \to G, \\
&(g, u).a = \lambda^+_g \lambda^-_{u^{-1}}(a) = p^-_r(auga^{-1})ag^{-1}.
\end{aligned}
$$

The (local) left dressing action for Λ_- is given by

$$
\begin{aligned}
&G_+ \times (G_-)^{\mathrm{op}} \times G \to G, \\
&(g, u).a = \lambda^-_g \lambda^+_{u^{-1}}(a) = p^-_r(ag^{-1}u^{-1}a^{-1})au. \qquad \square
\end{aligned}
$$

6.11. Remark. The dressing actions of G_+ on G_-, and of G_- on G_+ can also be reconstructed from this scheme. For $(g, u) \in G_+ \times G_-$ they are given by restricting the (local) actions λ^+, λ^-, ρ^+, and ρ^- of G on G appropriately (see 6.9):

$$
\begin{array}{ll}
\lambda^+_u(g) := p^+_r(gu^{-1}g^{-1})g, & \lambda^-_g(u) := p^-_r(ug^{-1}u^{-1})u, \\
\rho^+_u(g) := gp^+_l(gu^{-1}g^{-1}), & \rho^-_g(u) := up^-_l(ug^{-1}u^{-1}).
\end{array}
$$

Note that in these formulae one should replace, say, $p^+_r(gu^{-1}g^{-1})g = p^+_r(gu^{-1})$ only if the action is complete, or only for g and u near e, since the left-hand side is defined for all g and for u near e, whereas the right-hand side needs both g and u near e.

6.12. Corollary. *The dressing actions of G_+ on G_-, and of G_- on G_+ are (local) Poisson actions.*

Proof. We prove it only, say, for the left dressing action of G_- on G_+. At least locally this is given by

$$
\lambda^+ : G_- \times G_+ \to G_+, \qquad \lambda^+(u, g) = \lambda^+_u(g) = p^+_r(gu^{-1}),
$$

Due to Theorem 5.4(5) the mapping

$$
\tilde{\varphi} : G_+ \times G_- \ni (g, u) \mapsto gu^{-1} \in G
$$

is a Poisson mapping $(G_+ \times G_-, \Lambda^{G_+} \times \Lambda^{G_-}) \to (G, \Lambda_-)$. By Corollary 5.5 the (local) projection $p_r^+ : (G, \Lambda_-) \to (G^+, -\Lambda^{G_+})$ is a Poisson mapping, so the composition $\lambda^+ = p_r^+ \circ \tilde{\varphi}$ is also Poisson. □

7. Examples

7.1. Example. Let us assume that in the Manin decomposition $\mathfrak{g} = \mathfrak{g}_+ \oplus \mathfrak{g}_-$ the subalgebra $\mathfrak{g}_-$ is commutative then the simply connected Lie group G is isomorphic to the cotangent bundle $T^*G_+ \cong G_+ \rhd\!\!< \mathfrak{g}_-$, the semidirect product of G_+ and the dual Lie algebra, which is complete with respect to the dressing actions. $\varphi : G_+ \times \mathfrak{g}_- \to T^*G_+$ is the left trivialization, ψ is the right trivialization. This situation was described in detail in our earlier paper [2].

7.2. Example. We consider $\mathfrak{g}_+ = \mathfrak{su}(2)$ with the standard matrix basis

$$e_1 = \frac{1}{2}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \qquad e_2 = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad e_3 = \frac{1}{2}\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix},$$

satisfying $[e_1, e_2] = e_3$, $[e_2, e_3] = e_1$, and $[e_3, e_1] = e_2$. The following commutation rules $[e_1^*, e_2^*] = e_2^*$, $[e_1^*, e_3^*] = e_3^*$, and $[e_2^*, e_3^*] = 0$ for the dual basis in $\mathfrak{g}_- = \mathfrak{g}_+^*$ make $\mathfrak{g} = \mathfrak{g}_+ \oplus \mathfrak{g}_-$ into a Lie bialgebra which is isomorphic to $\mathfrak{sl}(2, \mathbb{C})$ as six-dimensional real algebra with $\mathfrak{g}_- = \mathfrak{sb}(2, \mathbb{C})$, where the elements of the dual basis are given by

$$e_1^* = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad e_2^* = \begin{pmatrix} 0 & -i \\ 0 & 0 \end{pmatrix}, \qquad e_3^* = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

The invariant symmetric pairing can be recognized as

$$\gamma(A, B) = 2\,\mathrm{Im}\,\mathrm{tr}(AB).$$

We consider now the double Lie group $G = SL(2, \mathbb{C})$ with $G_+ = SU(2)$ and $G_- = SB(2, \mathbb{C})$. We will write the elements as follows:

$$G = SL(2, \mathbb{C}) \ni a = \begin{pmatrix} z_1 & z_2 \\ z_3 & z_4 \end{pmatrix}, \quad \text{where } z_i \in \mathbb{C},\ z_1 z_4 - z_2 z_3 = 1,$$

$$G_+ = SU(2) \ni g = \begin{pmatrix} \alpha & -\bar{\nu} \\ \nu & \bar{\alpha} \end{pmatrix}, \quad \text{where } \alpha, \nu \in \mathbb{C},\ |\alpha|^2 + |\nu|^2 = 1,$$

$$G_- = SB(2, \mathbb{C}) \ni u = \begin{pmatrix} t & \gamma \\ 0 & t^{-1} \end{pmatrix}, \quad \text{where } t > 0,\ \gamma \in \mathbb{C}.$$

We define $\Lambda_+(a) = \frac{1}{2}(ra + ar)$ with $r = \sum_i e_i^* \wedge e_i$ on $SL(2, \mathbb{C})$ as explained in 5.1. We then extend it onto the whole space $GL(2, \mathbb{C})$ of all invertible matrices by admitting $a \in GL(2, \mathbb{C})$. Since the left and right invariant vector fields on $\mathfrak{gl}(2, \mathbb{C}) \cong \mathbb{C}^4 \cong \mathbb{R}^8$ satisfy the same commutation rules as their restrictions to $SL(2, \mathbb{C})$, we will get a Poisson structure.

Of course it is tangent to $SL(2,\mathbb{C})$, so that, if we consider the Poisson brackets between all matrix elements z_i and $\bar{z}_i$, the functions $\det = z_1z_4 - z_2z_3$ and $\overline{\det} = \overline{z_1z_4} - \overline{z_2z_3}$ will be Casimirs for the bracket. Thus we get a Poisson structure on $GL(2,\mathbb{C})$ whose restriction to $SL(2,\mathbb{C})$ is exactly Λ_+. We calculated the following Poisson brackets, which were also obtained independently by Zakrzewski [36].

$$
\begin{aligned}
&\{z_1, z_2\} = -\tfrac{1}{2}\mathrm{i} z_1 z_2, && \{z_2, z_3\} = \mathrm{i} z_1 z_4,\\
&\{z_1, z_3\} = \tfrac{1}{2}\mathrm{i} z_1 z_3, && \{z_2, z_4\} = \tfrac{1}{2}\mathrm{i} z_2 z_4,\\
&\{z_1, z_4\} = 0, && \{z_3, z_4\} = -\tfrac{1}{2}\mathrm{i} z_3 z_4,\\
&\{z_1, \bar{z}_1\} = -\tfrac{1}{2}\mathrm{i}|z_1|^2 - \mathrm{i}|z_3|^2, && \{z_2, \bar{z}_2\} = -\tfrac{1}{2}\mathrm{i}|z_2|^2 - \mathrm{i}|z_1|^2 - \mathrm{i}|z_4|^2,\\
&\{z_3, \bar{z}_3\} = -\tfrac{1}{2}\mathrm{i}|z_3|^2, && \{z_4, \bar{z}_4\} = -\tfrac{1}{2}\mathrm{i}|z_4|^2 - \mathrm{i}|z_3|^2,\\
&\{z_1, \bar{z}_2\} = -\mathrm{i} z_3 \bar{z}_4, && \{z_2, \bar{z}_3\} = \tfrac{1}{2}\mathrm{i} z_2 \bar{z}_3,\\
&\{z_1, \bar{z}_3\} = 0, && \{z_2, \bar{z}_4\} = -\mathrm{i} z_1 \bar{z}_3,\\
&\{z_1, \bar{z}_4\} = \tfrac{1}{2}\mathrm{i} z_1 \bar{z}_4, && \{z_3, \bar{z}_4\} = 0.
\end{aligned}
$$

The lacking commutators may be obtained from this list if we remember that the Poisson bracket is real, e.g. $\{\bar{z}_i, \bar{z}_j\} = \overline{\{z_i, z_j\}}$. One can then check that indeed det and $\overline{\det}$ are Casimir functions, and that $z_1 \leftrightarrow z_4$, $z_2 \mapsto -z_2$, and $z_3 \mapsto -z_3$ defines a symmetry of the bracket associated to the inverse $a \mapsto a^{-1}$ in $SL(2,\mathbb{C})$.

Our double group is complete since we have the following unique (Iwasawa) decompositions, where

$$
\varphi^{-1} : SL(2,\mathbb{C}) \to SU(2).SB(2,\mathbb{C}), \quad \text{where } s = \frac{1}{\sqrt{|z_1|^2 + |z_3|^2}},
$$

$$
\begin{pmatrix} z_1 & z_2 \\ z_3 & z_4 \end{pmatrix} = \begin{pmatrix} s z_1 & -s\bar{z}_3 \\ s z_3 & s\bar{z}_1 \end{pmatrix} \begin{pmatrix} 1/s & s(\bar{z}_1 z_2 + \bar{z}_3 z_4) \\ 0 & s \end{pmatrix},
$$

$$
\psi^{-1} : SL(2,\mathbb{C}) \to SB(2,\mathbb{C}).SU(2), \quad \text{where } t = \frac{1}{\sqrt{|z_3|^2 + |z_4|^2}},
$$

$$
\begin{pmatrix} z_1 & z_2 \\ z_3 & z_4 \end{pmatrix} = \begin{pmatrix} t & t(z_1\bar{z}_3 + z_2\bar{z}_4) \\ 0 & 1/t \end{pmatrix} \begin{pmatrix} t\bar{z}_4 & -t\bar{z}_3 \\ t z_3 & t z_4 \end{pmatrix}.
$$

Therefore, the bracket $\{\ ,\ \}$ is globally symplectic on $SL(2,\mathbb{C})$. This bracket is projectable on the subgroups $SU(2)$ and $SB(2,\mathbb{C})$, and for the 'left trivialization' $SL(2,\mathbb{C}) = SU(2)\cdot SB(2,\mathbb{C})$ it gives us the Poisson–Lie brackets on $SU(2)$:

$$
\begin{aligned}
&\{\alpha, \bar{\alpha}\} = -\mathrm{i}|\nu|^2, && \{\nu, \bar{\nu}\} = 0,\\
&\{\alpha, \nu\} = \tfrac{1}{2}\mathrm{i}\alpha\nu, && \{\bar{\alpha}, \bar{\nu}\} = -\tfrac{1}{2}\mathrm{i}\bar{\alpha}\bar{\nu},\\
&\{\alpha, \bar{\nu}\} = \tfrac{1}{2}\mathrm{i}\alpha\bar{\nu}, && \{\bar{\alpha}, \nu\} = -\tfrac{1}{2}\mathrm{i}\bar{\alpha}\nu,
\end{aligned}
$$

and on $SB(2,\mathbb{C})$:

$$
\{\gamma, t\} = \tfrac{1}{2}\mathrm{i}\gamma t, \qquad \{\bar{\gamma}, \gamma\} = \mathrm{i}\left(t^2 - \frac{1}{t^2}\right).
$$

It is possible to linearize $SB(2, \mathbb{C})$ with this Lie–Poisson structure. The mapping

$$\begin{pmatrix} t & \gamma \\ 0 & 1/t \end{pmatrix} \mapsto (\log(t), \operatorname{Re}\omega, \operatorname{Im}\omega),$$

where

$$\omega = \sqrt{\frac{R^2 - \log^2(t)}{|\gamma|^2}} \cdot \gamma, \qquad R = \frac{1}{2}\operatorname{arcosh}\left(\frac{|\gamma|^2 + t^2 + 1/t^2}{2}\right)$$

gives us a Poisson diffeomorphism between $(SB(2, \mathbb{C}), \Lambda^{G_-})$ and the linear Poisson structure defining the coadjoint bracket on $\mathfrak{su}(2)$, namely $z\partial_x \wedge \partial_y + y\partial_z \wedge \partial_x + x\partial_y \wedge \partial_z$. These formulae were first obtained by Xu, see also [36].

Since $H^2(SL(2, \mathbb{C})) = 0$, the symplectic structure $\omega = \Lambda_+^{-1}$ is exact, so there is a potential Θ with $d\Theta = \omega$. Moreover, $(SL(2, \mathbb{C}), \Lambda_+)$ is symplectomorphic to $T^*SU(2)$ with the canonical symplectic structure, since $(G_- = SB(2, \mathbb{C}), \Lambda^{G_-})$ is Poisson equivalent to $\mathfrak{sb}(2, \mathbb{C})$ with its $\mathfrak{su}(2)$-dual Poisson structure. So from the Poisson point of view there is no difference between $(SL(2, \mathbb{C}), \Lambda_+)$ and $T^*SU(2)$ (they are isomorphic as symplectic groupoids), but the group structures differ.

7.3. Example. On the '$ax + b$' Lie algebra $\mathfrak{g}_+$ spanned by X_1, X_2 with commutator $[X_1, X_2] = X_2$ the cobracket given by $b'(X_1) = 0$ and $b'(X_2) = X_1 \wedge X_2$ defines a Lie bialgebra structure. The Lie bracket on $\mathfrak{g}_- = \mathfrak{g}_+^*$ is then given by $[Y_1, Y_2] = Y_2$, and the remaining commutator relations on $\mathfrak{g} = \mathfrak{g}_+ \oplus \mathfrak{g}_-$ is given by $[X_1, Y_1] = 0$, $[X_1, Y_2] = -Y_1$, $[X_2, Y_1] = X_2$, $[X_2, Y_2] = -X_1 + Y_1$. A matrix representation of $\mathfrak{g}$ is the Lie algebra $\mathfrak{gl}(2, \mathbb{R})$ via

$$X_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad X_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad Y_1 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad Y_2 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$

with the metric

$$\gamma(A, B) = \operatorname{tr}(AJBJ), \quad \text{where } J = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

The subgroups $G_\pm$ of the Lie group $G = GL^+(2, \mathbb{R})$ of matrices with determinant > 0 are given by

$$G_+ = \left\{ \begin{pmatrix} x & y \\ 0 & 1 \end{pmatrix} : x > 0 \right\}, \qquad G_- = \left\{ \begin{pmatrix} 1 & 0 \\ a & b \end{pmatrix} : b > 0 \right\},$$

The calculation of the affine Poisson tensor Λ_+ on G in the coordinates

$$\begin{pmatrix} x & y \\ a & b \end{pmatrix} \quad \text{gives } \Lambda_+ = xy\partial_x \wedge \partial_y + ab\partial_a \wedge \partial_b + xb(\partial_x \wedge \partial_b + \partial_a \wedge \partial_y).$$

It is degenerate at points with $xb = 0$ and vanishes at $x = b = 0$. This shows that $G_+G_- \neq G$. Indeed, one can easily see that G_+G_- consists of all matrices with $b \neq 0$, and G_-G_+ of those with $x \neq 0$. This should mean that the dressing vector fields are not all complete. Indeed, $\lambda_+(X_1) = -x(bx - ya)\partial_x$, which restricted to G_+ gives $-x^2\partial_x$, a vector field on $\mathbb{R}^+$ which is not complete since its flow is given by

$$\mathrm{Fl}_{x_0}^{-x^2\partial_x}(t) = \frac{x_0}{tx_0 + 1}.$$

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